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DYNAMICALLY DEVELOPED FORCE AT THE BOW OF AN ICEBREAKER

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DYNAMICALLY DEVELOPED FORCE AT THE BOW OF AN ICEBREAKER

by

RODERICK MACLEOD WHITE

United States Coast Guard Academy, B.S.
(1950)

Massachusetts Institute of Technology, Nav. E.
(1956)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1965

Signature of Author ______________________________
Department of Naval Architecture and
Marine Engineering, May 14, 1965

Certified by ______________________________
Thesis Supervisor

Accepted by ______________________________
Chairman, Departmental Committee on Graduate Students

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Biographical Note

Roderick M. White was graduated from the United States Coast Guard Academy in 1950. He was awarded a Bachelor of Science Degree and was commissioned in the United States Coast Guard.

During the period from 1950 to 1953 the author served on a weather patrol ship, first as a deck officer and then later gained qualification in engineering. During the first part of 1953 he attended Merchant Marine Safety School and was assigned to Merchant Marine inspection in Boston for a relatively brief period.

From 1953 to 1956 he studied Naval Architecture and Marine Engineering at Massachusetts Institute of Technology. An M.I.T. Fellowship was awarded to him and as a result he studied shipbuilding in Copenhagen, Denmark during the summer of 1955. In June, 1956 he was awarded full membership in the Society of the Sigma Xi and was graduated with the degree of Naval Engineer.

For a period following graduation from M.I.T. he worked in the Production Department of the United States Coast Guard Yard in Curtis Bay, Maryland. The first portion of the time was spent as "Ship Superintendent" for ship repair; the latter portion of the time he was in charge of construction of 95-foot patrol coats.

From 1957 to 1959 the author served as Engineer Officer of the U.S. Coast Guard Icebreaker "Westwind". During that time he made four extended ice cruises (in the Antarctic as well as the Arctic). He also gained qualification as an officer-of-the-deck which enabled him to handle the ship in ice on many occasions.

In 1959 Lieutenant Commander White commenced a tour of duty as a member of the faculty of the Department of Applied Science and Engineering at the United States Coast Guard Academy. In 1962 he was appointed to the permanent commissioned teaching staff. He has taught Naval Architecture and most of the undergraduate courses in Applied Science.

In addition to membership in Sigma Xi he is a member of the Society of Naval Architects and Marine Engineers, and a member of the American Society for Engineering Education.

DYNAMICALLY DEVELOPED FORCE AT THE BOW
OF AN ICEBREAKER

by

RODERICK MACLEOD WHITE

Submitted to the Department of Naval Architecture and Marine Engineering on 14 May 1965, in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

The object of this research has been to develop a suitable mathematical model (computer program) for the prediction of the dynamically developed force under the bow of an icebreaker during (or resulting from) encounter with virtually unyielding ice.

The selection of characteristics for polar icebreakers has been primarily based on experience. Some analytical work has been done on uninterrupted progress (steady icebreaking). Essentially there has been only one analysis of the force resulting from ramming, which represents the primary maximum capability of a polar icebreaker. The validity of that particular dynamic analysis is doubtful because of the use of undefined losses and an improper resolution of the impact (crushing phase). No other approach has been made, until now, to the dynamic aspects of icebreaking.

This solution is based primarily on Newton's Laws of motion. The problem was broken down into two basic phases. The crushing phase represents the local crushing of the ice to accommodate the bow. The sliding phase represents the sliding-up of the bow without further crushing. The final state represents (temporary) equilibrium when motion has stopped; the vertical force at the bow at this state is relatively sustained and is the most effective in breaking the ice.

The predictions of ship motions, as well as the forces, are produced by the computer program. These predictions have been compared with observed motions of a full scale polar icebreaker and have been found valid.

As a result of studying the effect on the downward force of the various characteristics of a polar icebreaker, the following selections and uses are recommended if greater downward force is to be attained:

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As a result of studying the effect on the downward force of the various characteristics of a polar icebreaker, the following selections and uses are recommended if greater downward force is to be attained:

- Large displacement
- High impact velocity
- Small bow angle
- Small spread angle complement (blunt bow)
- Small block coefficient
- Large waterplane coefficient
- High beam-to-draft ratio
- Low kinetic friction

It is vital to realize that the selection of characteristics to improve downward force leads (in almost all cases) to a worsening of the thrust requirement for extraction.

In order to reduce the problem of extraction, without reducing the downward force, the following selections are recommended:

- Low static friction
- High backing bollard thrust
- Small spread angle complement

Model tests having dynamic similitude may be run using geometrically similar models with a Froude Number equal to that of the ship at impact. It is necessary that the model "ice" have a compressive failure stress equal to that of the ice divided by λ.

Thesis Supervisor:____
Jacob P. DenHartog

Title: Professor of Naval Architecture and Mechanical Engineering

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The support and guidance of the Thesis Committee has been greatly appreciated. The members and their respective specialty concerning this research are as follows:

Jacob P. DenHartog, Professor of Mechanical Engineering and Naval Architecture (Dynamics)

Martin A. Abkowitz, Professor of Naval Architecture (Ship Motions)

Clyde M. Adams, Jr., Professor of Metallurgy (Ice)

Justin E. Kerwin, Associate Professor of Naval Architecture (General)

The guidance and encouragement of the Graduate Registration Officer has also been appreciated.

Philip Mandel, Professor of Naval Architecture

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Special appreciation, for patience and encouragement, is due my wife.

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I INTRODUCTION

General

There exists today an increasing need for icebreakers, icebreakers which are well designed on the basis of good technology as well as experience. There is now an abundance of experience to rely on. However, relatively speaking, there is a shortage of analytical thought and understanding of the basic mechanics of icebreaking. More work along these lines is desired and needed.

Icebreakers can be defined as vessels which are specifically designed to break ice. Frequently they serve many other purposes but for the sake of definition it is best to keep in mind that the primary function is to be able to break ice. Furthermore, icebreakers can be categorized in many different ways. In the simplest sense let us divide them into two categories, polar and sub-polar. It is to be implied from this that the polar icebreakers are for the real heavy-duty work. Operation of this type of icebreaker eventually means encountering ice which cannot be penetrated by the icebreaker. It may be because of rafting, where sheets of ice build on one another due to the pressures of wind, water, and/or ice or it may be that the ice is simply too thick and/or too strong. There can quite easily be the case where the ice is a monolithic sheet extending from shore to shore in which case even moderate thicknesses may be sufficient to stop progress. Snow covering can make penetration similar fighting one's way through a room full of pillows. At any rate, any

polar icebreaker will eventually face the day when it will not be able to force its way through the ice - or has already faced that day!

Maximum Capability

Naturally it is very desirable to attain the greatest capability for a given investment. What is this greatest capability? It can be measured in many different ways depending upon the purpose for which a given ship is intended. Capability for a passenger ship would be measured by different standards from those of a tanker. Likewise the most important criterion for a successful polar icebreaker is different from most ships; it is primarily the ice it is able to break through. Other items naturally take on importance too such as the breadth of the channel formed and even the size of the broken pieces of ice left in its wake. Most important though is its ability to impart a relatively sustained force to the ice in the vertical direction.

For illustration let it be assumed that two polar icebreakers exist, icebreakers A and B. Assume that they each are about the same general size, have similar propulsion means, and represent equal investments. Each of them performs an identical mission . Each of them can open up a harbor in Greenland in late spring. Each of them succeeds in escorting supply ships to Artic or Antartic bases. Each of them is costing about the same to operate and each is earning its keep, so to speak. In other words, up to this point each is performing its mission. Then one day they are assigned to the task of opening a polar harbor in mid spring. The ice

conditions are severe. Icebreaker A can break through even though it is necessary to ram the ice, back off and ram again, breaking away large sections each time. Even though the progress is not smooth and steady, there is progress and icebreaker A accomplishes its mission. On the other hand, icebreaker B approaches these same ice conditions and it is found that, even by ramming, icebreaker B cannot break off any ice and makes no progress. Icebreaker B has failed in this particular mission. It is not a question of partly succeeding in this mission; it is simply a question of success or failure.

One may say that icebreaker A was designed better since it obviously performed better. What made its design better? It was able, under ramming conditions, to develop a greater downward force under the bow. It may seem obvious but it must be stated that since the illustrated ice conditions, whatever they may have been, were identical, the difference in the ability was inherent in the ship.

If one were to design an icebreaker at this point he would naturally duplicate or improve the design of icebreaker A, thereby quite rightfully utilizing the experience gained. Along with this it would be desirable to understand *why* icebreaker A was better. To do this it would be necessary to understand the mechanics of what is happening. If one were to analyze the mechanics of the problem then it would be possible to predict the best performance for a given investment.

Note that the thickness of ice to be broken is not a necessary part of the answer. As any who have been engaged in icebreaking know, a given icebreaker may be able to break through ten feet of ice in one location where there is not complete coverage or where there may be some deterioration. The same icebreaker may not be able to break through five feet of ice the same day in a different location where there may be complete coverage and the ice may be land-fast. So it is quite misleading to indicate that a certain icebreaker can break through a certain number of feet of ice. To compare the ability of one icebreaker to another it is of much more value to state what magnitude of relatively sustained downward force can be generated at the bow as a result of ramming. Although other items are of importance also, ultimately the most important answer lies in that value.

Parameters

It is necessary to determine how the parameters involved effect this answer, the downward force resulting from ramming. The problem is complex and is a function of the form of the ship, the displacement, the thrust, the location of the center of gravity, physical properties of ice, and perhaps other variables. It would seem that the angle the stem makes relative to the ice and the angle of spread of the waterlines at the entrance would be important parameters of ship form. Some answers would appear obvious at first glance. Increasing thrust and displacement would increase this downward force, but to what degree do they effect it?

Ultimately other questions must be considered. Let it be assumed that the bow angle and other parameters are chosen such that a maximum downward force would exist. If the ice does not yield, except locally, the velocity will become zero, the ship will have reached its farthest point of sliding up on the ice. Is the static friction at this point so great that the ship cannot back off? This is obviously an important consideration and limitation to free choice of parameters.

Table I lists most of the icebreakers constructed. Although not all of them are to be considered polar icebreakers, it is interesting to note the relatively large variation in the selection of parameters. In recent years there has been a strong tendency to set the bow angle at or near 30°. The bow angle is the angle from the design waterline to the stem. This choice is based on one, and only one, good reason; it has worked. It is interesting and significant to note that there has never been an analytical attempt to justify this choice.

Mechanics of Icebreaking; Terms

Some discussion of terms to be used is in order. The methods of breaking ice with an icebreaker can be expressed fundamentally in two ways. The bow of an icebreaker is sloped so that there is a downward component of force produced on the ice. It is this vertical component which is effective in breaking the ice since the ice is significantly more vulnerable to a force applied in this direction, particularly when sustained. The horizontal component, even the horizontal wedging action is no where

TABLE I CHARACTERISTICS OF VARIOUS ICE-BREAKERS

Name	Murtaga	Stadt Reval	Isbrytaren	Trouvor	Nadeshnij	Gaidamak	Sampo	Baikal	Apu	Ermack	Tarmo	Feodor Litke	Chief Wawatam
Home port	Helsingfors	Reval	Gothenburg	Kronstadt	Vladivostok	Bizerte	Helsingfors	Lake Baikal	Helsingfors	Leningrad	Helsingfors	Archangel	Muskegon Mich.
Year built	1890	1895	1895	1896	1897	1898	1898	1899	1899	1899	1907	1909	1911
Where built	Stockholm	Stettin	Gothenburg	Copenhagen	Copenhagen	Kiel	Newcastle	Newcastle	Kiel	Newcastle	Newcastle	Barrow-in Furness	Toledo, O.
CHARACTERISTICS													
Gross tonnage, registered	676	576	568	999	1212	724	1339	...	568	4955	1574	2375	2990
Net tonnage, registered	...	43	290	176	258	99	91	...	326	2357	173	930	1793
Length, overall	156'-0"	148'-8 1/2"	139'-9 1/2"	166'-0"	192'-0"	198'-7"	202'-0"	290'-0"	144'-0"	320'-0"	220'-1"	265'-0"	338'-10"
Waterline length	137'-5 1/2"	147'-3 3/4"	134'-0"	161'-5"	180'-0"	163'-0"	191'-6"	...	141'-9 1/4"	310-7"	210'-5"	250'-5"	...
Maximum beam	36'-0 o/3"	38'-11"	35'-6"	40'-3"	42'-11"	42'-0"	43'-0'	57'-0"	35'-6"	71'-6"	47'-0 1/2"	47'-11"	62'-0"
Waterline beam	35'-4"	37'-1"	39'-0"	42'-0"	41'-6"	42'-0"	42'-0"	...	34'-0"	70'-1 1/2"	46'-5"	47'-6"	...
Depth, molded to weather deck	24'-11"	18'-7 1/2"	...	22'-3"	24'-5"	18'-0"	29'-5"	28'-6"	...	42'-6"	...	26'-6"	20'-9"
Normal draft, d	15'-7"	14'-0"	18'-10"	18'-9"	17'-9"	12'-2"	18'-3"	...	18'-0"	24'-0"	18-'2"	16'-9"	...
Maximum draft	18'-1"	...	...	...	21'-0"	16'-0"	...	...	...	28'-0"	...	22'-6"	...
Normal displacement, tons	825	865	720	1450	1700	1030	1850	4200	800	7875	2300	2570	...
Maximum displacement, tons	900	1600	1300	2000	2170	1065	...	...	900	10,000	...	4600	...
Stem angle to waterline	33°	30°	31°	25°	30°	30°	2.2°	...	20°	22°	20°	52°/70°	...
Angle of normal to bow plating and ℄ plane													
Flare amidships at waterline	10°	...	11°	...	...	...	...	...	...	20°	...	3 1/2°	...
Complement	28	...	21	38	6	39	43	...	23	112	43	...	35
Speed, knots	12.5	11.5	12/12.6	9.5/13	10/14.4	11.5/13	10/13	...	12/16	14	13	18.5	...
HULL COEFFICIENTS													
Block	0.382	0.396	0.410	0.430	0.443	0.438	0.441	...	0.323	0.527	0.454	0.452	...
Midship													
Waterplane													
Longitudinal													
L.O.A./B, maximum	4.33	3.82	3.94	4.12	4.48	4.73	4.70	5.09	4.06	4.48	4.68	5.53	...
L.W.L./B, waterline	3.89	3.97	3.94	4.14	4.29	3.93	4.56	...	4.17	4.43	4.53	5.26	...
d/B, waterline	0.441	0.378	0.554	0.481	0.422	0.296	0.435	...	0.529	0.342	0.391	0.353	...
$\Delta/(\frac{L}{100})^3$	317.4	270.7	299.2	344.9	291.5	237.8	263.4	...	280.8	262.9	246.9	164.5	...
PROPELLING MACHINERY	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.
No. shafts	1	1	1	1	1	1	2	3	1	4	2	2	3
No. forward	0	0	0	0	0	0	1	1	0	1	1	0	1
No. aft	1	1	1	1	1	1	1	2	1	3	1	2	2
FORWARD ENGINES													
Horsepower,[a] normal	...	...	...	...	...	...	1200	...	...	2500	1300	...	...
Horsepower,[a] forced	...	...	...	...	...	...	1417	...	...	3000	1660	...	...
R.P.M.	...	...	...	...	...	...	90/97	...	...	...	89/102	...	...
AFTER ENGINES													
Horsepower,[a] normal	1200	1600	1200	2000	2475	1300	1300	...	1350	7500	2200	7000	...
Horsepower,[a] forced	1793	...	1449	2600	3530	1940	1635	...	1600	9000	2190	7580	...
R.P.M., normal/maximum	83	...	102	110	95/106	83/94	80/86	...	84/90	105	89/102	125	...
Total horsepower, normal	1200	1600	1200	2000	2475	1300	2500	...	1350	10,000	3500	7000	4500
Total horsepower, forced	1793	...	1449	2600	3530	1940	3052	...	1600	12,000	3850	7580	...
Cruising radius	2300	...	1100/1750	1100/2200	1560/2040	...	1404	...	...	10,650	1250	1665/2775	...
Ballast pumps	1	1	1	1	1	1	1	...	...	1	...	1	...
Total capacity, tons per hr.	1150	1000	...	1000	1000	1000	700	...	...	600	...	250	...
Capacity of trimming tanks	...	...	...	400	4000	...	...	...	...	502	...	...	...
Thickness of ice belt plating, in.	...	...	...	...	...	...	...	...	...	1 1/4"	...	...	...
Frame spacing	...	...	...	...	...	...	...	...	...	12"	...	...	...
LCF aft of FP													
BM_L													

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a - Indicated horsepower unless otherwise stated

TABLE I(cont) CHARACTERISTICS OF VARIOUS ICE-BREAKERS

Name	Jaakarhu	N.B. McLean	Store Bjorn	Ymer	Gota Lejon	J.Stalin	Sisu	Raritan	Cactus	Storis	North-wind	Macki-naw	Canadian Car Ferry
Home port	Helsing-fors	Ottawa, Canada	Copen-hagen	Stock-holm	Gothen-burg		Helsingfors	Philadelphia, Pa.	Boston, Mass.	Boston, Mass.	Boston, Mass.	Cheboygan, Mich.	Charlottetown, P.E.I.
Year built	1926	1929	1931	1932	1933	1937	1938	1939	1941	1941	1944	1944	1945
Where built	Rotterdam	Halifax	Aalborg	Malmo	Gothen-burg	Leningrad	Helsing-fors	Bay City, Mich.	Duluth, Minn.	Toledo, O.	San Pedro, Cal.	Toledo,	Sorel, Que.
CHARACTERISTICS													
Gross tonnage, registered	2622	3254	1393	3053	1355	...	1610	...	...	...	...	...	7000
Net tonnage, registered	...	1171	509	535	151	...	...	...	...	...	...	...	4500
Length, overall	263'-0"	277'-0"	196'-10"	257'-10 5/8"	183'-8 3/4"	350'-0"	210'-6"	110'-0"	180'-0"	230'-0"	269'-0"	290'-0"	372'-6"
Waterline length	246'-0"	260'-0"	180'-5 1/4"	246'-0 1/4"	173'-10 3/4"	335'-0"	194'-9"	105'-0"	170'-0"	220'-0"	250'-0"	280'-0"	348'-0"
Maximum beam	63'-1"	60'-4"	49'-4"	63'-4"	49'-5"	76'-0"	47'-6"	26'-5"	37'-0"	43'-0"	63'-6"	74-4"	61'-0"
Waterline beam	60'-2 1/2"	...	46'-10 3/4"	61'-0 1/2"	46'-11"	74'-6"	46'-6"	25'-0"	35'-0"	41'-0"	62'-0"	70-0"	...
Depth, molded to weather deck	31'-10"	31'-0"	23'-11 1/2"	33'-0"	28'-2 5/8"	30'-0"	18'-6"	14'-11 5/8"	17'-4 3/4"	19'-3 1/2"	37'-9 1/2"	28'-0 3/4"	24'-9"
Normal draft, d	21'-10"	19'-6"	18'-2 1/2"	21'-0"	20'-0"	26'-3"	16'-0 1/2"	10'-6"	12'-0"	14'-0"	25'-9"	19'-0"	19'-0"
Maximum draft	...	...	10-0 1/4"	...	...	29'-8"		11'-0"	12'-8"	14'-10"	29'-1"	...	...
Normal displacement, tons	4850	5034	2500	3465	1900	9300	2000	328.6	940	1760	5300	5140	...
Maximum displacement, tons	4900	...	...	4350	...	11,000	...	354	1025	1864	6515	...	...
Stem angle to waterline	25°-30°	...	24°	25°	23°	30°	24°	30°	30°	30°	30°	30°	...
Angle of normal to bow plating and ℄ plane						47.8							
Flare amidships at waterline	20°	...	11°	18°	20°		...	20°	20°	20°	20°	20°	...
Complement	47	...	45	42	30	142	100	16	47	103	145	132	...
Speed, knots	13.5	...	13.5	14/15.87	...	15.5	16	12.3	13	14	16	16	16.5
HULL COEFFICIENTS													
Block	0.525	...	0.568	0.384	0.408	0.498	0.482	0.417	0.457	0.478	0.465	0.497	...
Midship	0.789	...	0.848	0.801	...	...	...	0.728	0.817	0.835	0.752	0.812	...
Waterplane	0.741	...	0.734	0.725	...	...	...	0.738	0.728	0.761	0.724	0.728	...
Longitudinal	0.645	...	0.628	0.593	...	...	...	0.578	0.558	0.572	0.618	0.612	...
L.O.A./B, maximum	4.17	4.59	3.99	4.07	3.75	4.61	4.43	4.16	4.87	5.35	4.24	3.90	6.11
L.W.L./B, waterline	4.09	...	3.85	4.03	3.71	4.50	4.18	4.20	4.86	5.37	4.03	4.00	...
d/B, waterline	0.363	...	0.388	0.344	0.426	0.352	0.345	0.420	0.343	0.341	0.415	0.271	...
$\Delta/(\frac{L}{100})^3$	325.8	286.4	425.6	232.6	361.3	247.4	270.8	283.9	191.3	165.3	339.2	234.1	...
PROPELLING MACHINERY	Steam recip.	Steam recip.	Steam recip.	Diesel electric	Steam recip.	Steam recip.	Diesel electric	Diesel electric	Diesel electric	Diesel electric	Diesel electric	Diesel electric	Diesel electric
No. shafts	3	2	3	3	2	3	3	1	1	1	3	3	4
No. forward	1	0	1	1	1	...	1	0	0	0	1	1	2
No. aft	2	2	2	2	1	...	2	1	1	1	2	2	2
FORWARD ENGINES													
Horsepower,[a] normal	2530	...	1800	3000 shp	1300	...	1335 shp	...	...	...	3333/0	3333/0	5600/2400
Horsepower,[a] forced	3000	...	...	3300 shp	1500	...	...	...	...	...	...	...	...
R.P.M.	130	...	125	155	140	...	160	...	...	...	140-210	175-200	...
AFTER ENGINES													
Horsepower[2], normal	5310	6500	3600	6000 shp	2500	10,000	2670 shp	1000 shp	1000 shp	1800 shp	6666/ 10,000	6666/ 10,000	5600/ 7200
Horsepower,[a] forced	6000	...	...	6600 shp	2500	...	...	...	...	...	...	...	...
R.P.M., normal/maximum	117	105	100	140	120	125	140	236	170[b]-208[c]	160[b]-195[c]	105[b]-145[c]	136[b]-170[c]	...
Total horsepower, normal	7840	6500	5400	9000 shp	3800	10,000	4005 shp	1000 shp	1000 shp	1800 shp	10,000 shp 10,000	10,000/ 10,000	11,200/ 9600
Total horsepower, forced	9000	...	...	9900 shp	4000	...	...	...	...	...	...	...	...
Cruising radius	2950	...	...	...	...	...	...	1800	10,000.	10,000	10,800 full power	10,800 full power	...
Ballast pumps	7	1	...	1	1	4	...	1	1	1	3[d]	4[d]	...
Total capacity, tons per hr.	1900	900	...	200	1800	1755	...	57	17	17	9344[d]	12,459[d]	...
Capacity of trimming tanks	750	...	...	330	283	1791	150	20	22	72	717	1541	...
Thickness of ice belt plating, in.	...	...	...	1"	...	1"	1"	3/4"	3/4"	7/8"	1 5/8"	1 5/8"	...
Frame spacing	...	...	...	24"	...	20"	15 3/4"	20"	21"	21"	16"	16"	15"

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a Indicated horsepower unless otherwise stated

b Ice-breaking or towing

c Running free.

d Heeling pumps

TABLE I (cont'd) CHARACTERISTICS OF VARIOUS ICE-BREAKERS

Name	Petr Velikij	Sainte Marie	Suur Tool	Isbry taren II	Pollux	Mikula Selianinovitch	Leonid Krasin	Stepan Makarow	Lenin	Vioma	Isbjorn	Krisjanis Valdemars	Atle
Home port	Riga (Sunk in 1915)	Muskegon, Mich.	Reval	Stockholm	French (mine layer)	Cherbourg	Kron-stadt	Arch-angel	Kron-stadt	Helsing-fors	Copen-hagen	Riga	Stock-holm
Year built	1912	1913	1914	1914	1915	1916	1917	1917	1917	1917-1924	1923	1925	1926
Where built	Gothenburg	Toledo, O.	Stettin	Stock-holm	Newcastle	Montreal	New-castle	New-castle	New-castle	Helsing-fors	Copen-hagen	Dalmuir	Gothen-burg
CHARACTERISTICS													
Gross tonnage, registered	1267	2383	2417	1651	1613	3165	5105	2372	3828	1510	978	1932	1777
Net tonnage, registered	427	1620	822	329	653	2042	2246	885	1310	...	358	757	257
Length, overall	182'-1"	267'-4"	247'-4 1/2"	200'-0"	210'-0"	292'-0"	323'-3"	248'-0"	281'-0"	210'-7"	170'-7"	196'-6"	204'-7"
Waterline length	170'-7"	252'-0"	236'-4 1/4"	190'-7"	198'-1 1/2"	275'-0"	297'-0"	236'-0"	273'-0"	200'-10"	156'-2"	185'-0"	194'-7"
Maximum beam	51'-0"	62'-0"	57'-1"	55'-10"	50'-6"	57'-5"	71'-0"	57'-0"	63'-10"	46'-7"	40'-2 1/2"	55'-10"	55'-9"
Waterline beam	48'-6 5/8"	54'-8"	56'-1 1/4"	52'-6"	...	..	70'-6 1/2"	...	...	45'-10 7/8"	39'-4 1/2"	54'-0"	53'-2"
Depth, molded to weather deck	27'-5"	25'-0"	22'-10"	28'-8 1/2"	27'-6"	32'-0"	41'-4 1/2"	30'-4"	31'-11"	18'-9"	21'-0"	28'-6"	28'-9"
Normal draft, d	19'-1"	14'-0"	18-7 5/8"	20'-8 5/8"	19'-10"	19'-3"	26'-0 1/4"	22'-0"	19'-0"	16'-9"	18'-6"	22'-0"	19'-8"
Maximum draft	21'-4"	...	18'-9	21'-6"	...	...	30'-0 1/2"	...	20'-6"	...	20'-6"	...	20'-8"
Normal displacement, tons	1610	2567	3562	2350	3100	...	8730	4570	5074	2070	1330	2800	2464
Maximum displacement, tons	1953	...	3622	...	4830	...	10,620	4600	5620	2180	1670	...	2740
Stem angle to waterline	30°	...	18°	20°	...	...	25°	18°	18°	23°	20°	33°	24°
Angle of normal to bow plating and ℄ plane													
Flare amidships at waterline	20°	...	...	24°	...	...	20°	15°-20°	15°-18°	15°	...	17°	16°
Complement	60	27	65	41	100	90	190	...	118	44	30	48	45
Speed, knots	14.4	...	12/14	13.4	14	15.5	15	15	16	14	12.5	15.2	15.5
HULL COEFFICIENTS													
Block	0.357	0.479	0.505	0.397	...	...	0.562	...	...	0.469	0.409	0.446	0.424
Midship	0.708	...	...	0.755	...	...	...	...	...	...	...	...	0.733
Waterplane	0.723	...	...	...	...	...	...	...	...	...	...	...	0.662
Longitudinal	0.533	...	...	0.530	...	...	...	...	...	...	...	...	0.571
L.O.A./B, maximum	3.57	4.31	4.33	3.58	4.16	5.09	4.55	4.35	4.40	4.52	4.37	3.52	3.66
L.W.L./B, waterline	3.51	4.61	4.21	3.63	...	...	4.21	...	...	4.37	3.97	3.43	3.66
d/B, waterline	0.393	0.256	0.332	0.395	...	...	0.369	...	...	0.365	0.470	0.407	0.370
$\Delta/(\frac{L}{100})^3$	324.4	152.8	269.8	339.5	...	...	333.2	...	...	255.5	350.6	442.2	334.4
PROPELLING MACHINERY	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.	Steam recip.
No. shafts	2	2	3	2	2	2	3	3	3	2	1	2	2
No. forward	1	1	1	1	0	0	0	1	1	1	0	1	1
No. aft	1	1	2	1	2	2	3	2	2	1	1	1	1
FORWARD ENGINES													
Horsepower,^{a} normal	1000	...	1500	800	...	...	...	1550	2500	1100	...	1500	1400
Horsepower,^{a} forced	1255	...	2300	1200	...	...	...	1900	2600	1280	...	1500	2000
R.P.M.	220/286	...	120	220	...	...	...	115/140	...	146	...	263	85/105
AFTER ENGINES													
Horsepower,^{a} normal	2200	...	3000	1800	4000	8000	10,000	4000	5000	2600	2500	3700	2600
Horsepower,^{a} forced	2660	...	4600	2500	4300	...	...	4700	5400	2870	3000	4000	4000
R.P.M., normal/maximum	130/143	...	100	130	105	100	...	90/105	...	111	90	122	100/125
Total horsepower, normal	3200	2500	4500	2600	4000	8000	10,000	5550	7500	3700	2500	5200	4000
Total horsepower, forced	3915	...	6900	3700	4300	...	...	6600	8000	4150	3000	5500	6000
Cruising radius	1600	...	2920	1600	...	...	4250/8700	1600/4900	2150/3800	1625	...	...	2000
Ballast pumps	2	...	...	2 ballast 1 salvage	...	...	5	...	1	1	...	2	1
Total capacity, tons per hr.	2400	...	...	925	...	...	1000	...	700	500	...	1800	3600
Capacity of trimming tanks	700	...	...	...	...	...	...	...	...	...	...	...	...
Thickness of ice belt plating, inc.	...	11/16"	...	...	...	...	1 1/4"	...	1 1/4"	...	...	1"	...
Frame spacing		15"	...	...	...	...	12"	...	12"	...	...	12"	...

TABLE I (cont'd) CHARACTERISTICS OF VARIOUS ICE-BREAKERS

Name	Diberville Canada	Kapitan Class	Glacier	Lenin	MacDonald	[illegible]
Home port	Canada	USSR	USA	USSR	Canada	USSR
Year built	1952	1954	1955	19[illegible]	1960	1960-[illegible]
Where built	Lauzon	Helsinki	Pascagoula		Lauzon	Helsinki
CHARACTERISTICS						
Gross tonnage, registered						
Net tonnage, registered						
Length, overall	310'	275'	[illegible]	440'	31[illegible]'	390'
Waterline length	300'	[illegible]	[illegible]	420'	307'-3"	363'-4"
Maximum beam	[illegible]	[illegible]	[illegible]4'	90'-[illegible]"	70'	[illegible]
Waterline beam	[illegible]	[illegible]	72'-[illegible]"	90'	69	79'
Depth, molded	40'-3"	[illegible]'-2"	[illegible]		41'	46'
Normal draft						
Maximum draft	[illegible]'-3"	[illegible]	[illegible]	30'-[illegible]"	29'	34'-[illegible]"
Normal displacement, tons						
Maximum displacement, tons	[illegible],930	[illegible]	[illegible]	16,000	[illegible]	1[illegible],000
Stem angle to waterline	[illegible]°	[illegible]	[illegible]	[illegible]	[illegible]6	26°
Angle of normal to bow plating and ℄ plane, deg.			[illegible]°			[illegible]°
Flare amidships at waterline	[illegible]°	[illegible]°	[illegible]°		[illegible]°	[illegible]°
Complement	1[illegible]	[illegible]	[illegible]	200	77	
Speed, knots	14.[illegible]		[illegible]	[illegible]	[illegible]	
HULL COEFFICIENTS						
Block	0.[illegible]9		[illegible]	[illegible]	[illegible]	[illegible]
Midship	.[illegible]4					[illegible]
Waterplane			[illegible]			[illegible]
Longitudinal						
L.O.A./B, maximum						
L.W.L./B, waterline	[illegible]	[illegible]	[illegible]	4.[illegible]	[illegible]	[illegible]
d/B, waterline						
$\Delta \quad (\frac{L}{100})^3$						
PROPELLING MACHINERY	Steam Recip.	Diesel Elect.	Diesel Elect.	Nuclear Turbo-elect.	Diesel Elect.	Diesel Elect.
No. shafts						
No. forward	-	2	-	-	-	-
No. aft	2	2	2	[illegible]	2	3
FORWARD ENGINES						
Horsepower,^{a} normal						
Horsepower,^{a} forced						
R.P.M.						
AFTER ENGINES						
Horsepower^a, normal	10,800		21,000	[illegible]9,[illegible]00w		
Horsepower^a, forced						
R.P.M., normal/maximum	145/	120/	17[illegible]	[illegible]	130/170	
Total horsepower, normal	10,800	[illegible]	21,000	[illegible]9,[illegible]	15,000	[illegible]
Total horsepower, forced						
Cruising radius	12,000			[illegible]	20,000	[illegible]
Ballast pumps						
Total capacity, tons/hr.						
Capacity of trimming tanks						
Thickness of ice belt plating, in.						
Frame spacing						
LCF aft of FP			[illegible]'	[illegible]2[illegible] est		
BM_L			[illegible]'	[illegible] est		

This table is based on references (1),(2),(3). and (4).

near as effective since the structural shape of the ice is such that it can withstand tremendous forces in this direction. As an illustration of this, if one desired to break a pane of glass he would apply a force normal to the plane (causing a bending moment) rather than apply a force against the edge in the plane of the glass.

In the simpler condition of uninterrupted progress, steady state, the icebreaker moves along maintaining a relatively constant velocity. Except for minor variations it can be considered that there are no accelerations involved. The other fundamental method is ramming. This is where the icebreaker backs away from any contact with solid ice, proceeds forward so that there is a forward velocity at the time of initial contact and strikes the ice with its sloped bow. The bow rides up on the ice and a force is generated acting against the ice. Some of this force is the result of the thrust being applied by the propulsion; the rest of it is the result of converting the kinetic energy before impact to potential energy. The two methods then are uninterrupted progress, where any acceleration is negligible, and ramming, where the acceleration (negative) is extremely important. All actual icebreaking is done by one of these two methods or by something in between these two extremes. It should be apparent that ramming will lead to the greater force development since uninterrupted progress is, in a sense, a minimum ramming situation where the accelerations have reduced to zero.

Methods

Many methods of icebreaking have been utilized with varying degrees of success. The earliest account of deliberate icebreaking was contained in remarks made by a British Rear Admiral in 1865. (5)* "I have had two years experience in ramming the ice. Our vessels had long oblique overhanging stems to lift the bow over the ice. We struck the floe ice of about six feet in thickness, end-on; a man at the bowsprit end dropped down on the ice and placed a boarding pike as a mark where the blow was given. The vessel backed astern, and then ran directly for the mark which had been placed on the ice; the man who was standing by the crack thus made picked up his boarding pike and placed it on the edge of the crack, so that the vessel might be steered directly for it again, and the third time the ice opened and the steam tender towed the ship through; such was the constant practice.------ We ran the vessel's nose dead onto the ice and did the ice more injury than the vessel, for the vessel never was injured during several years of such service." It is interesting to note that this very first account was of ramming. Perhaps the object of this present research could be stated more succinctly in the words Belcher used, "Do the ice more injury than the vessel"!

Prior to breaking ice using iron-clads another method had been used on occasion. (6) Ships would become beset in harbor ice and it became necessary to free them. Men would be recruited from the city in gangs of fifty to two hundred and they would be equipped with pikes and saws. With

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* Number indicates literature citation of Appendix D.

these tools they would clear a path from the ship out to open water.

Even today one of the more obvious methods of getting from one side of an ice field to the other is used. Go around, or at least minimize the contact with ice by following leads in the ice field. Even the largest of icebreakers is operated with that discretion.

Another method has been used with some success, particularly on icebergs. Sections of the ice have been painted black using soot or some form of paint. These colored sections will absorb the heat very quickly and melting takes place relatively rapidly in these areas.

Although quite expensive, it has been found that firing torpedoes under the ice will either break it completely or at least make it relatively easy for an icebreaker to penetrate. (7).

Most polar icebreakers carry explosives to use on the ice. Their effect on hummocks or very solid ice is actually quite limited. However, when an icebreaker becomes stuck after ramming the ice an explosive charge may have the very beneficial effect of jarring the ship and the ice enough to allow the ship to break free from the grasp of static friction and back off.

The Russians have utilized streams of water at the bow at great pressure to break and destroy the ice. They have found this more suitable than torpedoes. (8)

One of the most unusual and interesting approaches to the problem has been tried by the Russians and was published in an official magazine of the U.S.S.R.. (7) "An underwater explosion accompanied by an exceptionally bright beam of light acts particularly strongly on ice. Very strong light, arising through water into ice, produces in it many tiny cracks. This was observed by the well-known English Physicist Tyndall more than a hundred years ago. The little cracks lower the solidity of the ice to such an extent that, passing through the ice after the light, the shock wave of the explosion relatively easily magnifies the cracks and destroys the ice."

Although seemingly irrelevant, other less scientific approaches have been tried. In the winter of early 1959 an icebreaker was attempting to escort a small ice-protected tanker into a harbor in Newfoundland. The ice was quite solid and reached from shore to shore across the bay. The icebreaker was repeatedly rammed into and onto the ice. Heeling tanks, trimming tanks, and explosives were used. After one full day of frustration the progress could easily be measured in inches. The Commanding Officer decided to stop and relax for the night. The tanker was brought up astern and all hands joined in one massive bingo game, which lasted for most of the night. When light appeared the following morning it was quickly noticed that during the night a lead had opened up all the way from the ship to the dock. The icebreaker and the tanker continued their trip without further opposition. However, it seems difficult to justify bingo

as a scientific method.

Certainly the most commonly used method for polar icebreaking is to use a well designed icebreaker with the bow sloped such that it is possible to generate a significant downward force under the bow, particularly as a result of ramming.

History of Icebreakers

Although 1851 is given as the first year a vessel was built specifically for breaking through ice, very little was ever published concerning it. Generally it is regarded that the first successful icebreaker built for that express purpose was constructed on 1871 and was named appropriately "Eisbrecher I". (9)

It was constructed for the purpose of keeping the channel open from Cuxhaven to Hamburg throughout the winter season. This icebreaker was 130 feet long and had an engine of 300 ihp. The concept of design and operation was then much the same as it is now. There was a sloping stem in order to get a downward component. It was intended that the vessel progress as constantly as possible, but when pack ice was encountered the icebreaker was to be backed down and then ram against the ice at full speed. It had a rather full bow with a sloping stem. This was frequently copies in the years to follow and later it was modified to a spoon-shaped bow. A disadvantage with the full spoon-shaped bow was quickly discovered; if there was snow on the ice it would pile up ahead and impede or stop progress. Small entrance angles and small stem

angles were advocated but not tried as many felt that this would be poor from a structural point of view. Propellers at the bow were also advocated at that time and were actually in use before the turn of the century. It is to be noted then that very large steps in thought and application were taken in the last thirty years of the last century.

Mathematical Expression of Icebreaking

In order to see what has been done concerning the prediction of the downward force under the bow, one need only to look back to the same period of time mentioned above. From then until the present only four men have left a deep impression by the development of a mathematical expressions for the mechanics of icebreaking. As will be seen, the first three did not develop an expression suitable for ramming; they developed equations for uninterrupted progress.

R. Runeberg was the first to analyze the mechanics of the icebreaking process. (10) Particularly considering that he was unable to base any of his work on previous developments, he did a remarkable amount. Some of this takes in the concept of ramming but unfortunately no useable equation for the downward force during ramming was developed.

Even in 1888 he recognized that "the vertical component should be as large as possible" since this does the breaking. His equation for the downward force is redeveloped completely in Appendix A.

His equation states that the downward force under the bow, F_{BZ}, for uninterrupted progress is a function of the following:

T_{IB} = Thrust available for icebreaking, LB.

i_B = Stem angle with reference to base plane, deg.

β = Angle with respect to the ℄ plane of a normal to the shell at the bow, deg.

f_k = Coefficient of kinetic friction between ice and hull.

The downward force is expressed as follows:

$$F_{BZ} = \frac{T_{IB}\,(\cos i_B \cos \beta - f_k \sin i_B)}{(\sin i_B \cos \beta + f_k \cos i_B)} \qquad (A12)^{*}$$

Runeberg suggests the use of 0.05 for f_k.

The following assumptions were used for the development of this equation:

1. There are no momentum effects.
2. The forward motion through the water is effectively non-existent so that the thrust can therefore all be applied to icebreaking.[**]
3. Thrust was directed horizontally at all times.[**]
4. The direction of friction force (along the direction defined by the slope of the stem) remained the same during forward horizontal progress.[**]
5. Trim, although it exists, is not great enough to affect the solution.[**]

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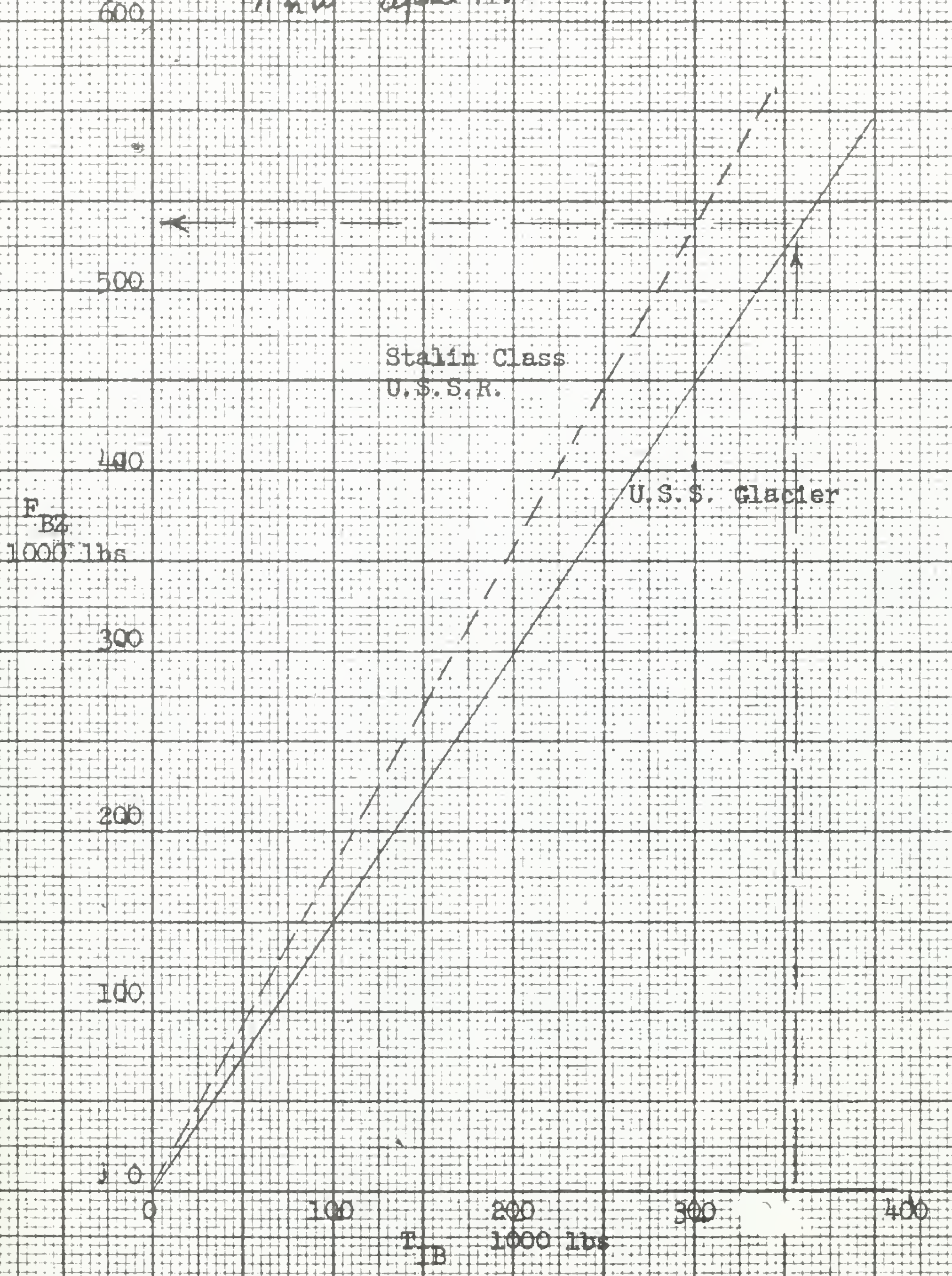
[*] Numbers in parentheses refer to equations of the appendix.

[**] These assumptions were used but not stated.

Figure I

Downward Force for Uninterrupted Progress vs Thrust based on Runeberg's Equation

Rhw april 1965



	U.S.S. Glacier	Stalin Class, U.S.S.R.
i_B	30.0°	25.0°
β	40.3°	42.2°
f_k	0.05	0.05

His equation was developed on the basis of the ship sliding up on the ice very slowly but it was intended to be used as a good approximation for an icebreaker making uninterrupted progress in the horizontal direction.

Figure I shows a plot of the downward force under the bow versus the icebreaking thrust based on Runeberg's equation for both the U.S.S. Glacier and the Stalin class of the U.S.S.R. For example, if a thrust useable for icebreaking of 358,400 lb were developed by the U.S.S. Glacier, a downward force of 537,600 lb would be generated under the bow. This represents the maximum available force downward for this given value of thrust.

A. Kari was the second to analyze the mechanics of the icebreaking process. (11) Both the statics and dynamics of icebreaking was discussed in his paper "The Design of Icebreakers", but his equations for the downward force under the bow are of use only for uninterrupted progress. A complete redevelopment of his equations for the downward force is given in Appendix A.

His equations state that the downward force under the bow, F_{BZ}, for uninterrupted progress is a function of the following:

Δ = Displacement, tons.

θ = Change of trim, deg.

L = Length between perpendiculars, ft.

and for equation (A25).

GM_L = Longitudinal metacentric height, ft.

or for equation (A26).

H = Draft, ft.

$$C = \frac{BM_L \times H}{L^2} = \frac{GM_L \times H}{L^2} = 0.07$$

The downward force is expressed as follows:

$$F_{BZ} = \frac{4480\ \Delta\ GM_L \sin\theta}{L} \qquad (A25)$$

and

$$F_{BZ} = \frac{4480\ \Delta\ C\ L \sin\theta}{H} \qquad (A26)$$

The following assumptions were used in the development of these equations:

1. There are no momentum effects.*
2. The vertical rise of the bow is equal to the thickness of the ice.* (This assumption fortunately has no bearing on the development of equations (A25) and (A26) but it is used later in his work to determine the thickness of ice which can be broken.)
3. The distance from the point of contact with the ice to the center of flotation is equal to the distance from the point of contact with the ice to the center of gravity.
4. The effective displacement is not affected by the force at the bow nor is the draft.*

---

* These assumptions were used but not stated.

5. As a result of 4, the center of flotation and the longitudinal metacenter remain fixed.*
6. $GM_{L_{\omega}} \cong GM_L$
7. The value of C is 0.07.
8. There is no frictional force.*

At one point in the development he set the summation of moments equal to zero but failed to do the same with the summation of forces. If he had done so a discrepancy would have been apparent. The equations were developed for an icebreaker having its bow slide slowly up onto the edge of the ice but he intended that the equations be used for an icebreaker making uninterrupted progress in the horizontal direction. At best they are a good approximation only for the stopped equilibrium position.

Figure II represents an illustrative plot of the downward force for uninterrupted progress versus the change in trim in degrees using Kari's equation (A26). Unless an arbitrary limit for the change in trim, θ, is given, the maximum force under the bow cannot be obtained from the equation directly. One would have to solve for it separately using an equation such as the one developed by Runeberg based on T_{IB}.

D. R. Simonson, a Coast Guard Lieutenant, was the third to analyze the mechanics of the icebreaking process. (12) The purpose of his work was to determine a bow profile which would represent an equilibrium condition

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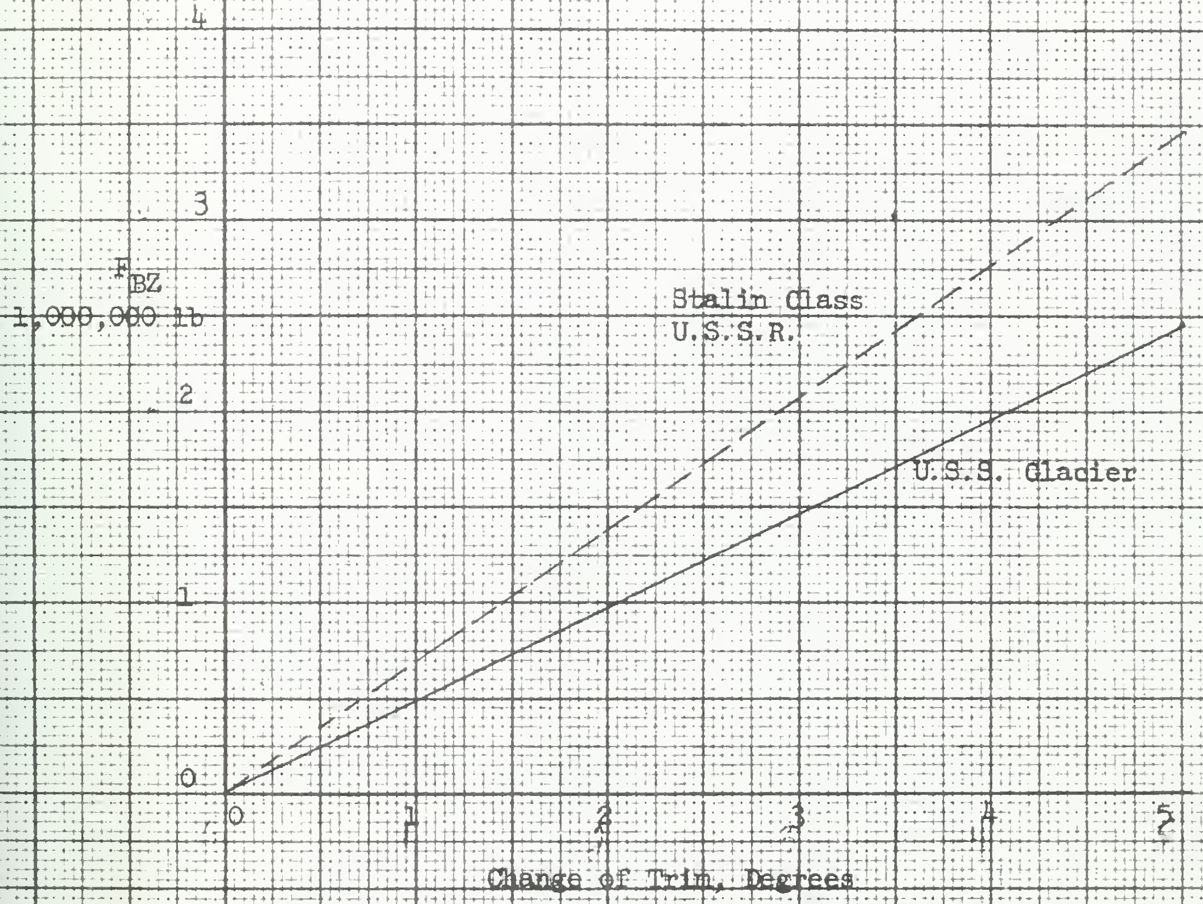
* These assumptions were used but not stated.

Figure II

Downward Force for Uninterrupted Progress vs Degrees Change of Trim based on Kari's Equation

RmW 4/65



	U.S.S. Glacier	Stalin Class U.S.S.R.
A	8,640 tons	11,000 tons
L	290 ft.	335 ft.
H	28 ft.	29.5 ft.
C	0.07	0.07

regardless of trim if the other factors were held constant. This lead to a mathematical description of the stem contour which turned out to be somewhat spoon-shaped. As a necessary step toward that determination he developed an equation for uninterrupted progress. A complete redevelopment of his work is given in Appendix A.

All of his work is statical since he felt that "momentum should be neglected as it is desirable to break ice without charging or ramming."

His equation states that the downward force under the bow, F_{BZ}, for uninterrupted progress is a function of the following:

T_{IB} = Thrust available for icebreaking, LB

i_B = Stem angle with reference to base plane, deg.

θ = Change of trim, deg.

The downward force is expressed as follows:

$$F_{BZ} = \frac{T_{IB}}{\tan (i_B \mp \theta)} \qquad \text{(A43)}$$

The following assumptions were used for the development of this equation:

1. There are no momentum effects.*
2. Friction with the ice is negligible.*
3. Thrust is directed horizontally at all times.*

---

*These assumptions were used but not stated.

4. The center of flotation serves as a pivot point.*

5. There is no change in displacement.*

Figure III represents an illustrative plot of the downward force for uninterrupted progress versus the icebreaking thrust based on Simonson's equation for both the U.S.S. Glacier and the Stalin Class of the U.S.S.R. For example, if a thrust useable for icebreaking of 358,400 lb were developed by the U.S.S. Glacier, a downward force of 595,000 lb would be generated under the bow. This represents the maximum available force downward for this given value of thrust. Note that it is necessary to solve for the trim independently or make a suitable assumption. The forces indicated in Figure III are all in excess of those indicated by Runeberg's equation which are illustrated by Figure I. This is due to the fact friction is neglected by Simonson.

In fact, Simonson's equation is limited to being a good approximation for the stopped equilibrium position, not really uninterrupted progress.

During 1946 a book was published in Russia entitled "Vessels for Artic Navigation" written by I. V. Vinogradov. (13) It contained the development of an equation for the downward force under the bow of an icebreaker which resulted from ramming. The work was significant in that it represented the first time that this force due to ramming was put into useable mathematical form.

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* These assumptions were used but not stated.

Figure III

Downward Force for Uninterrupted Progress vs Thrust based on Simonson's Equation

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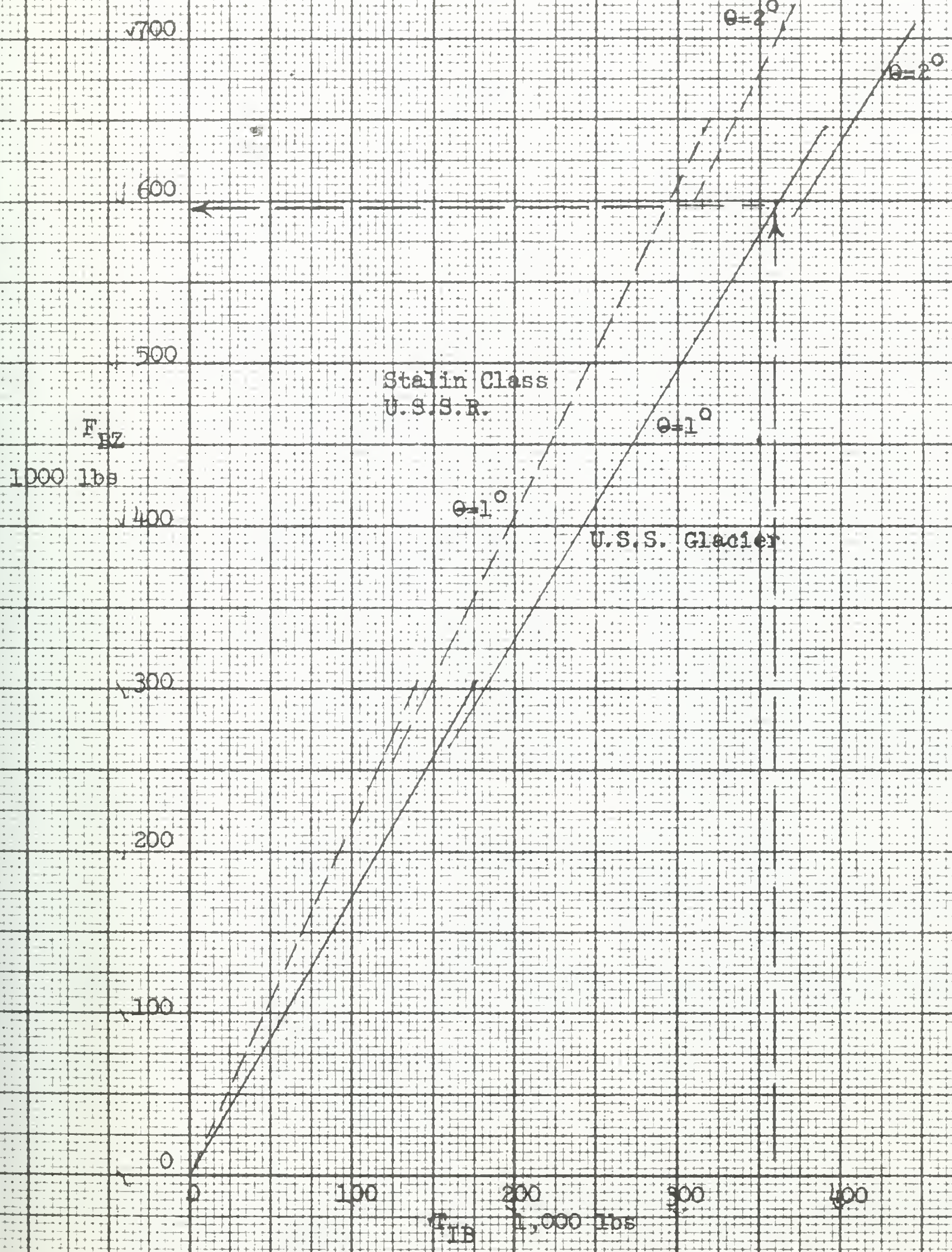

	U.S.S. Glacier	Stalin Class U.S.S.R.
i_B	30.0°	25.0°

L. W. Ferris has paraphrased Vinogradov as follows (14): "The analysis is based on the following concept: An icebreaker moving with known velocity strikes a uniform ice shelf and the bow of the ship glides up until the downward pressure reaches a magnitude which causes the ice shelf to collapse. While the ship is climbing the ice shelf, the propellers continue to push. In general, the forward motion of the ship is not reduced to zero at the instant when the ice collapses."

"The quantity which is to be determined is the maximum value of the vertical force P developed on the stem of the icebreaker."

Since ramming is taken into account by use of the principle of the conservation of energy, a much larger number of terms (i.e. parameters of ship form) will be necessary than have appeared in previously mentioned solutions for uninterrupted progress.

Vinogradov's equation states that the downward force under the bow, F_{BZ}, (P in his equations) for the ramming condition is a function of the following[*]:

F = Coefficient of sliding friction. (f_k)

φ = Angle of stem, deg. (i_B)

β = Angle of normal to shell plating with respect to ℄ plane, deg. (β)

δ = Block coefficient (δ)

---

[*] The symbols used are those of Vinogradov. Symbols in parentheses are those used commonly.

α = Waterline coefficient (α)

Q = L/2 plus the distance aft from to the center of flotation, ft. (L/2 - LCF)

L = Length between perpendiculars, ft. (L)

D = Draft, ft. (H)

M = Longitudinal metacentric height, ft. (GM_L)

T = Thrust, tons. (T)

W = Displacement, tons. (Δ)

E = Coefficient of resistution. (e)

V_o = Speed just prior to impact, ft/sec. (V_o)

V_1 = Speed while sliding up (normally taken as zero to get maximum P), ft/sec. (V_1)

This downward force under the bow is expressed as follows:

$$P = XT + \left\{ X^2T^2 + \frac{Y}{A} W^2 \cdot \frac{v_o^2\left[1-(1-e^2)\sin^2\varphi\right] - v_1^2}{g\ D} \right\} \qquad (A65)$$

where

$$X = \frac{1 - \frac{f}{\cos\beta}\tan\varphi}{1 + \frac{f}{\cos\beta}\cot\varphi}\cot\varphi \qquad (A63)$$

$$Y = \frac{1}{1 + \frac{f}{\cos\beta}\cot\phi} \qquad (A64)$$

$$A = \frac{\delta}{\alpha}\left[1 + \left(\frac{k_1}{k_2}\right)\frac{1}{4\alpha}\right] \qquad (A61)$$

$$k_1 = \frac{a}{(L/2)}$$

$$k_2^2 = \frac{m\,D\,\delta}{\alpha^2 L^2} \qquad k_2 = \frac{1}{\alpha L}\sqrt{m\,D\,\delta}$$

The following assumptions were used by Vinogradov in the development of this equation:

1. Thrust was directed horizontally at all times*.
2. Change in trim and draft do not seriously affect properties of the waterplane or the longitudinal metacentric height.
3. $GM_L \cong BM_L$

In addition to the assumptions listed above, many expedients were taken and these deserve some comment or criticism so that the reader may have a better idea of the validity of Vinogradov's equation. The most necessary of comments or criticisms follow:

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* This assumption was used but not stated.

1. Trim was taken into account during the solution for movement of the icebreaker (in order to determine the distance a force moved in doing work), but it is disregarded (or considered negligible) in the solution for the resultant perpendicular to the stem.
2. Thrust, T, was kept as a constant representing total thrust. There would be an improvement in the use of his equation if one were to consider this as the thrust applied to breaking ice, T_{IB}, since some of the total thrust is used in the resistance of the water.
3. Although kinetic energy and work are used for the basis of his work, there is no mention of the possibility of forces due to the acceleration. One of his key equations, (A54), sets the summation of forces equal to zero when in fact there is a large deceleration.
4. There is no mention of the fact that some of the kinetic energy while sliding up may be in the form of rotational energy as well as translational energy.
5. The change of trim is based on the original displacement using the equation for a couple when actually the effective displacement is changed.

6. Q is used exclusively as a constant representing the distance from the center of flotation to the forward perpendicular, which is assumed to be the original point of contact. For the geometrical determination of certain distances this is proper; however, it is not proper when this length is used as a moment arm from the point of contact. That particular distance is a variable. In fact, even using Vinogradov's equation as a basis, it can be shown (14) that the distance travelled after initial impact, for the U.S.S. Glacier ramming at six knots with $T = 160$ tons, is 28.7 feet. This means that Q should approximately decrease from the original 150 feet to 121 feet, a deccrease of approximately 19 o/o.

7. The expression for the loss of energy on impact is based on direct central impact. In other words, it is assumed that the loss is the same as if a perpendicular to the stem passed through the center of gravity of the icebreaker. (Later in the introduction much more can be found on this expedient. It is common to some other recent developments and comment will be reserved until these others have been mentioned.)

8. Although the sliding velocity is contained in the final equation, the equation is only valid when this sliding velocity, V_1, is equal to zero. This is not only because of the use of equilibrium in the solution, it is also because while sliding up there

is a component due to friction which is acting in opposition to the downward force, P. This term is not included in the final equation; it goes abruptly to zero as soon as the velocity goes to zero (and then it reverses).

9. In addition to the prohibition mentioned in 8, V_1 does not take on (in the equation) all values from V_o down to zero; this is due to the impact term which indicates that there is an immediate reduction,to some degree, of the velocity.

It must be noted that, in spite of the comments made above, Vinogradov's equation was the first equation that was of any use for the ramming condition. This ramming is quite important and for many years following 1946 this equation was by far the best criterion for the ability of an icebreaker. The development is given in Appendix A.

The result of a calculation which is given in the paper by Ferris (14) showed that this downward force for the U.S.S. Glacier ramming at six knots with a thrust of 160 tons became 3,225,600 pounds. This compares to 537,600 pounds (according to Runeberg) or 595,000 pounds (according to Simonson) for uninterrupted progress. These illustrative results are shown in Figure IV. It is quite readily seen that the order of magnitude of force generated by ramming completely overshadows the force generated during uninterrupted progress.

These four men named above (Runeberg, Kari, Simonson, and Vinogradov) have made the most significant contributions. Table II shows the

Figure IV

Comparison of Magnitudes of Force Developed Under Bow During Icebreaking (for the U.S.S. Glacier at 6 knots with T = 160 tons)

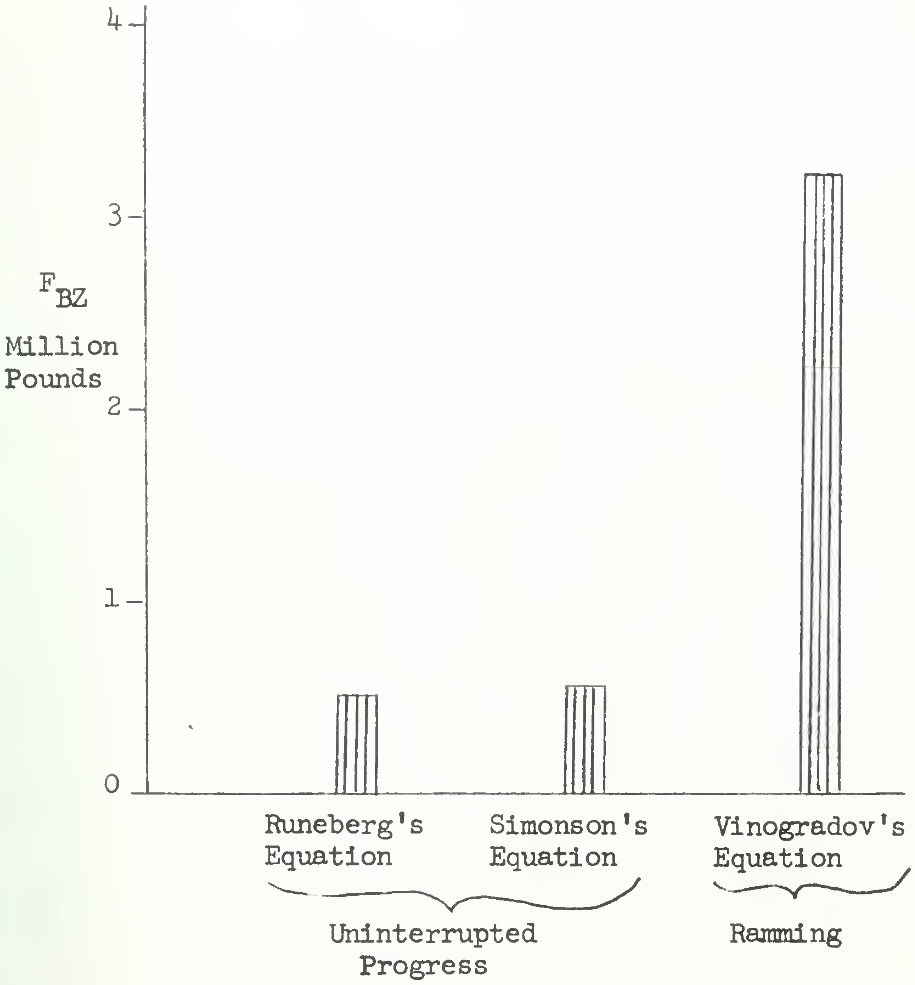
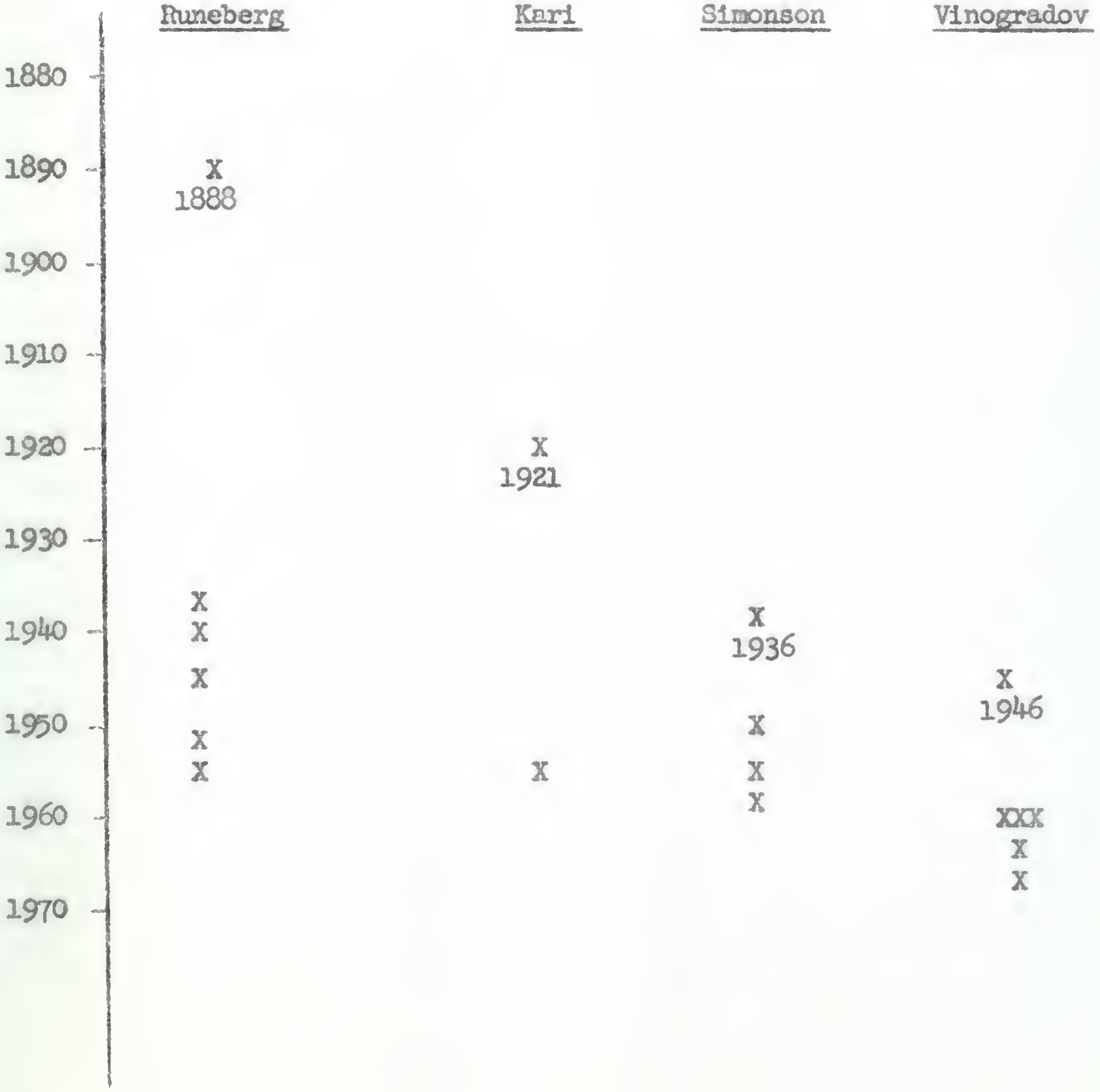


Table II

Use or Reference to Icebreaking Equations in

Relatively Prominent Publications Presented Chronologically



chronological popularity of their developments. Table III illustrates the parameters appearing in their equations, however, there have been others since then who have developed equations for icebreaking, but the equations lack the significance of those mentioned earlier.

In 1956 Jan-Erik Jansson presented an equation for the determination of work utilized in ramming of ice. (15) Unfortunately it does not include any equation for obtaining the downward force during icebreaking. However, his work is quite comprehensive and for that reason is included in Appendix A. He also uses the conservation of energy principle in his development. However, as a convenience he has disregarded loss at initial impact and has neglected friction.

In 1959 C. Richardson presented an equation for the downward force under the bow during ramming. (16) It was developed in conjunction with some model studies of the force system. The equation is similar to the equation presented by Vinogradov and is presented in Appendix A. The development was almost identical to Vinogradov's but did modify some of his weaknesses to some extent. For example, Richardson uses a term for the loss of energy due to wave and frictional resistance (not ice) from the instant of contact up to the moment the ice breaks or motion ceases. He also recognizes an effective increase in the mass of the icebreaker due to entrained water. For the most part, however, he has used the same assumptions and expedients that Vinogradov used and for that reason comments expressed earlier also apply here.

Table III

Parameters Appearing in the Equations of Downward Force by Various Developers

Parameter	Runeberg	Kari	Simonson	Vinogradov
	Developer			
T_{IB}	X		X	X
i_B	X		X	X
β	X			X
f_k	X			X
Δ		X		X
θ		X	X	
L		X		X
GM_L		X		X
δ				X
α				X
LCF				X
H				X
e				X
V_o				X
V_1				X

Symbols used are those given in Appendix C, Symbols and Their Titles.

As part of a report released in 1959 concerning the feasibility of a nuclear icebreaker, an equation concerning the relative magnitude of force "transmitted to the ice" at the bow was presented (12). This concerns uninterrupted progress only; it does not have to do with ramming. However, the development is given in Appendix A since it is of interest. The force under consideration here is the force perpendicular to each side of the bow. This could, of course, be resolved into the downward component, but this would lead right back to Simonson's equation since this work is based on that equation. It is interesting to note though that the equations of this report are based on the assumption that thrust remains parallel to the base line at all times, and not simply horizontal.

In 1962 V. R. Milano presented his modification of Vinogradov's equation (18). One of the main contributions was to express thrust as a function of "Bollard Pull". He also rewrote the equation so that the displacement may be solved based on other parameters including the desired downward force. The equation is given in Appendix A.

All four equations for ramming presented above are based on the principle of the conservation of energy. There is, of course, nothing wrong with the concept; any shortcomings exist only in the developments. Although the situation is obviously dynamic, which is the reason for the use of energy in the calculations, each at some point in his development, uses static equilibrium. They have set the summation of forces at a point equal to zero when there is acceleration involved.

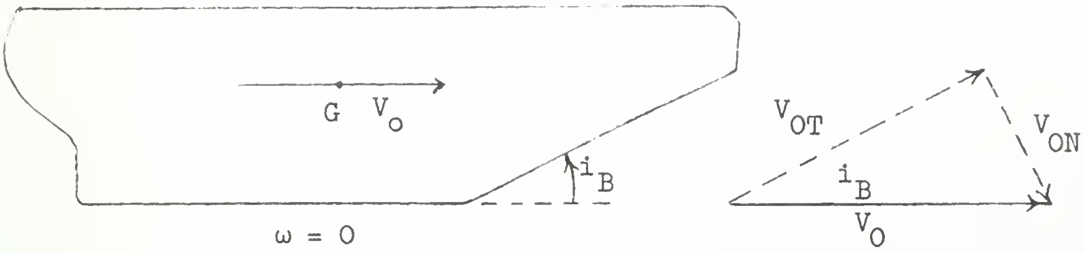
Furthermore, all developments of the ramming condition force utilize a very important term for the loss of energy at initial impact. (An exception to this is the development by Jansson; he neglects this term.) This loss of energy is determined using the coefficient of restitution, E. As this term has been developed, it is fundamentally in error. The use in their developments means that the impact has been direct central impact. This means that a normal to the stem at the point of contact would have to pass through the center of gravity (which it does not) and no rotation would be imparted. Furthermore, it means that the velocity component tangent (parallel to the inclination of the stem) to the stem is conserved on initial contact. The velocity component normal to the stem inclination would be reversed and be of magnitude equal to (E) X (initial normal component velocity). In most illustrations E has been set equal to 0.90 or 0.95. Figure V shows the implication of the acceptance of this form of impact energy loss. If one is willing to believe the energy-loss-at-initial-impact term, then one must also be willing to believe that after initial contact the velocity of the center of gravity is only slightly less in magnitude and is upward in direction at an angle which is almost twice the angle of the stem. Even one who has never seen an icebreaker in action would find this hard to believe.

Incidentally, carrying this concept further for the sake of illustration, it can be shown that the term would imply that a ship with a vertical stem hitting the ice at 10.0 knots would bounce so that it ended up going

Figure V

Implication of Acceptance of
Impact Energy Loss in Presently-
Used Ramming Equations

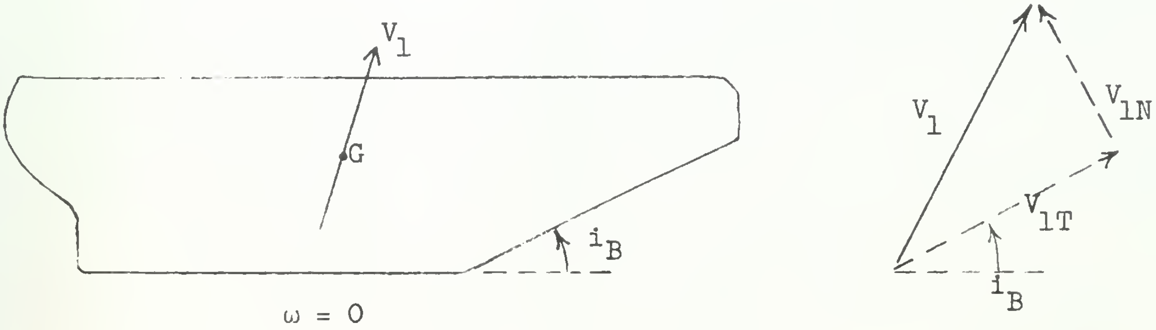
Before Initial Contact



If e = 0.90 $\qquad V_{IN} = -0.90\ V_{ON}$

$$V_{IT} = V_{OT}$$

After Initial Contact



astern at 9.0 or 9.5 knots. It is quite apparent this is not the case when a ship encounters ice.

Need for Suitable Analysis

It is interesting to note that many do not yet take full advantage of the equations which do exist. Many references (19), (20), (9), (21), (4), (22), and (23)., indicate the choice of the "optimum bow" or "standard bow" without any further mention of procedure or justification. Some of these simply state that 30° is the best angle.

Presently the downward force resulting from ramming is being used by some. In spite of any weaknesses which exist in the ramming equations, they are the best available and certainly present a more meaningful value than the force developed during uninterrupted progress.

Four criteria are presently used for measuring icebreaking capability. (4) These are listed by Lank as follows:

1. "Probably the most complete analysis of the action of an icebreaker in breaking a uniform sheet of ice is the one developed by Vinogradov." The downward force developed by ramming is apparently the most important criterion.
2. "A rough measure of the ability of a ship to force its way into leads or broken ice is the ratio of horsepower to beam."
3. "The horsepower displacement ratio has been widely used for comparing the relative power of icebreakers but probably does little more than express relatively ability to accelerate."

4. "For large icebreakers of generally similar hull form, the simple value of thrust at zero speed is probably as good a measure of forcing ability as any."

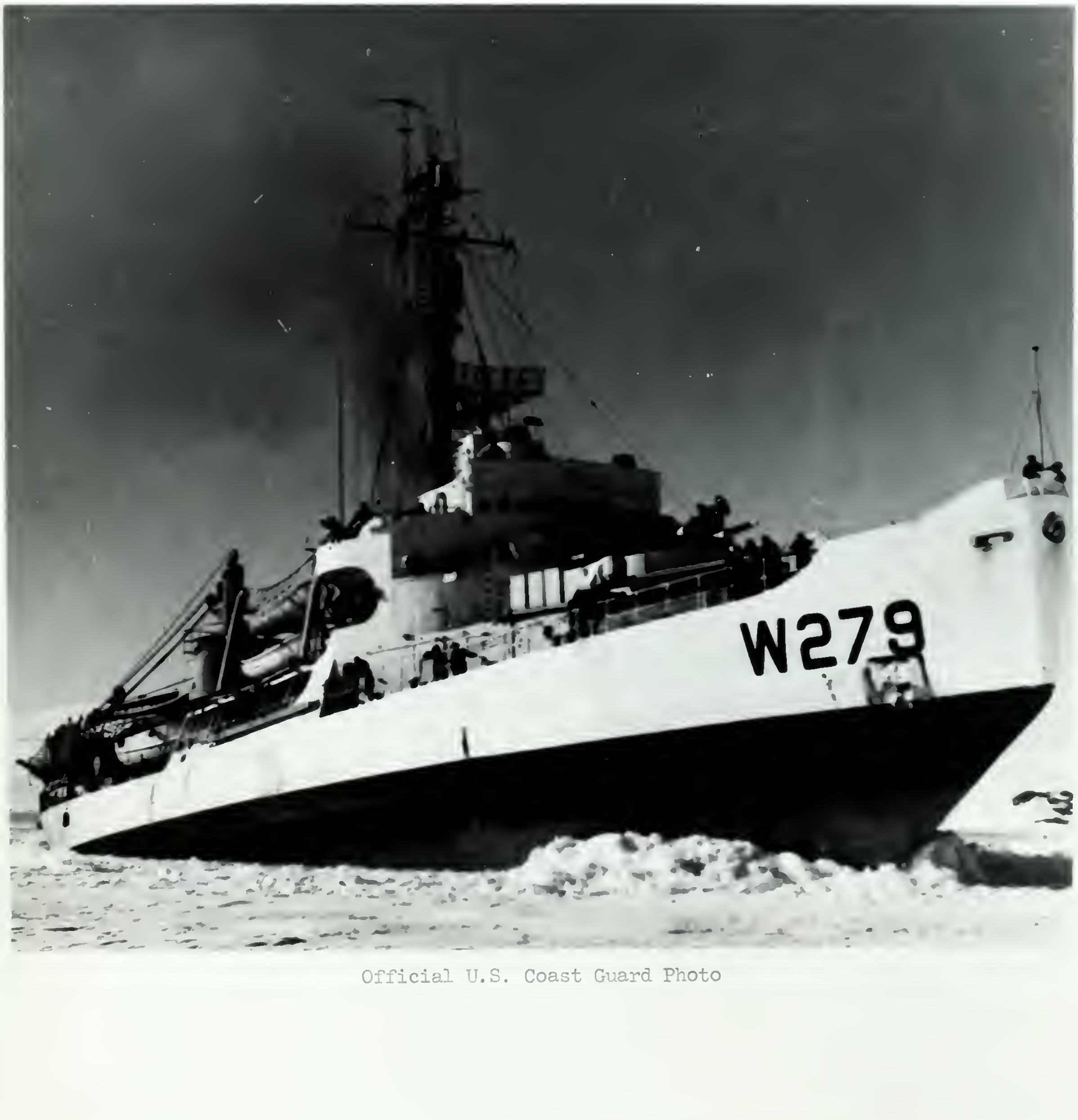
In general it is conceded that the equations for the ramming give the best measure of ability of a polar icebreaker. Just as generally, it is pointed out that more work must be done along these lines since the present equations do not quite lead to a proper representation.

Admiral E. H. Thiele said, in 1959, "icebreakers are relatively expensive to build and maintain. Every effort to make an icebreaker more effective through improvements in design and operating technique will be repaid many times".

The object of this research is to make icebreakers "more effective through improvement in design". The design can be improved by improving the measure of the most important criterion for a polar icebreaker, the downward force generated by ramming. Specifically, the object is to develop a suitable equation for the prediction of the dynamically developed force at the bow of an icebreaker during encounter with virtually unyielding ice.

Figure VI

"Wind Class" Icebreaker Ramming Ice



Official U.S. Coast Guard Photo

II Procedure

General

When a polar icebreaker encounters very heavy ice, the icebreaker must resort to *"ramming". The object of this technique is to get a large, relatively sustained, downward force under the bow. It is this sustained force which tends to cause the ice to collapse. See Figure VI and VII.

The bow first crushes into the ice until the bow is accomodated and supported sufficiently to allow sliding. The bow rises up to a point where forward progress ceases and the icebreaker settles at this point. It is as if the icebreaker were "grounded" at the bow.

The problem is to predict this downward force (F_{BZ}) as a function of the following parameters:

L = Length between perpendiculars, ft. (BP)

B = Waterline beam, ft. (B)

H = Normal draft, ft. (H)

Δ = Normal displacement, lb. (DIS)

i_B = Angle from base line to stem, radians (BA)

β = Angle of normal to bow plating with respect to the centerline plane, radians. (SA)

v_1 = Velocity of icebreaker immediately prior to initial contact, ft./sec.

α = Waterplane coefficient, dimensionless. (AL)

LCF = Distance from amidships to center of flotation (+ if forward, - if aft), ft. (CF)

LCG = Distance from amidships to center of gravity (+ if forward, - if aft), ft. (CG)

KG = Height of center of gravity above keel, ft. (GK)

d = Height of thrust line above base line, ft. (D)

T_{BOL} = Bollard thrust for rpm of sustained approach velocity, lb. (TB)

GM_L = Longitudinal metacentric height, ft. (GM)

f_k = Coefficient of kinetic friction between ice and ship, dimensionless. (FK)

σ = Compressive failure stress of ice, lb/ft^2. (SIG)

f_s^* = Coefficient of static friction, dimensionless (FS)

(Note: The symbols to the right in parentheses are those used in the Fortran computer program.)

The complete step-by-step solution is given in the appendix.

Definition of States and Phases

"State 1" is defined as the state of the icebreaker immediately prior to initial contact.

The "Crushing Phase" is that period when the ice is crushing locally to accomodate the bow. The bow is tending to rise and the ship is tending to slow down.

---

* This is not necessary for the downward force but is used for the extraction thrust.

"State 2" is defined as the state of the icebreaker when local crushing has ceased and the bow has a velocity tangent to the ship-ice interface. (In other words, there is no more penetration into the ice.)

The "Sliding Phase" commences at State 2. The bow slides up on the ice without further appreciable penetration. It is assumed the point of contact is fixed relative to the ice.

"State 3" occurs when the velocity of a point on the bow relative to the ice becomes zero. This does not necessarily imply that all velocities ($\dot{x}$, $\dot{z}$, and $\dot{\theta}$) are zero.

"State 4" occurs when all velocities have become zero and the icebreaker is in static equilibrium. The downward force under the bow, F_{BZ4}, is the relatively sustained force which is the object of this research.

The coordinates may be seen in Figure B-V.

Bow Forces During Crushing

It is assumed that all forces acting on the bow from the ice act at the intersection of the stem and the waterline. There are three forces acting at the bow. There is a force normal to the plating and it is assumed that this normal force may be represented by the product of the area of contact and the compressive failure stress of the ice. There is a component of friction force acting parallel to the stem in the plane of the plating. During crushing there is another friction force acting perpendicular to the stem in the plane of the plating. See Figure B-I.*

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* These figures appear in Appendix B.

As seen in Figures B-III and B-IV, these forces may be expressed as components in the x-direction and the z-direction.

$$F_{BXC} = N\,(\cos\beta + f_k \sin\beta)\sin(i_B + \theta) + N f_k \cos(i_B + \theta) \qquad \text{(B8)}^{+}$$

$$F_{BZC} = N\,(\cos\beta + f_k \sin\beta)\cos(i_B + \theta) - N f_k \sin(i_B + \theta) \qquad \text{(B3)}$$

where N is the force normal to the bow plating.

As mentioned earlier, the normal force N is the product of the area of contact due to penetration and the compressive failure stress of the ice.

$$N = \sigma A = \frac{\sigma}{\sin\beta\tan(i_B + \theta)}\left[x\tan(i_B + \theta) - \left(\frac{L}{2} - LCG\right)\theta + z\right]^2$$

Newton's Laws of Motion During Crushing

Figure B-IX illustrates the free body diagram of the icebreaker during the crushing phase. It is assumed that the icebreaker may be treated as a "solid body".

The forces acting on the bow are included in the equations of motion. Dropping negligible terms, the forces may be expressed as a function of x^2.

[+] The numbers in parentheses refer to equations in Appendix B.

As a result of summing forces in the x-direction,

$$\ddot{x} + a_1\dot{x} + a_2x^2 + a_3 = 0 \qquad \text{(B29)}$$

where a_1, a_2, and a_3 are constants representing the influence of parameters such as T_{BOL}, v_1, σ, i_B, β, f_k, and m_x. ("m_x" will be explained later. It is sufficient for the present to say that it is the mass of the icebreaker.)

Furthermore, by summing forces in the z-direction,

$$\ddot{z} + a_1\dot{z} + a_2z + b_1\theta + c_1x^2 = 0 \qquad \text{(B31)}$$

where a_1, a_2, b_1, and c_1 are constants representing the influence of parameters such as T_{BOL}, v_1, σ, i_B, β, f_k, m_z, and k_h. ("k_h" is a heave damping coefficient which will be explained later.

By summing moments around the center of gravity, by linearizing, by dropping negligible terms, and by substitution,

$$\ddot{\theta} + a_1\dot{\theta} + a_2\theta + d_1x^2 = 0 \qquad \text{(B51)}$$

where a_1, a_2, and d_1 are constants representing the influence of parameters such as I_θ (mass moment of inertia, to be explained later), k_p (pitch damping coefficient, to be explained later), Δ, and GM_L.

Solution for x During Crushing

It is natural to solve for x first. It is independent of other variables and the solutions for z and θ depend on x.

By dropping negligible terms,

$$\ddot{x} = \frac{-k_1}{m_x} x^2 \qquad \text{(B39)}$$

where k_1 is a constant incorporating the influence of σ, i_B, β and f_k.

By substitution, manipulation, and integration,

$$\dot{x} = \left(v_1^2 \frac{-2k_1}{3 m_x} x^3\right)^{1/2} \qquad \text{(B40)}$$

where the constants are as previously defined.

The equation for $\dot{x}$ cannot be directly integrated. Therefore a series expansion is used. From this it is possible to integrate to find t in terms of x. Negligible terms are then dropped. The expression can be x expressed as a function of t by using a reversion of the series. Retaining significant terms,

$$x = v_1 \left(t \frac{-k_1 t^3}{12 m_x}\right) \qquad \text{(B41)}$$

where k_1 and m_x have been previously defined.

Now that x has been found as a function of t, $\ddot{x}$ and $\dot{x}$ may also be expressed as functions of t by means of substitution.

Solution for θ During Crushing

The summation of moments can be expressed in terms of θ and t by the substitution of equation (B41).

The solution becomes

$$\theta = e^{\alpha_1 t} (A_1 \cos \beta, t + A_2 \sin \beta, t) + \frac{d_1}{c_1} t^2 - \frac{2b_1 d_1 t}{c_1^2} - A. \qquad (B53)$$

where α_1, β_1, A_1, A_2, b_1, d_1, and c_1 are constants reflecting the influence of I_θ, k_p, Δ, GM_L, L, LCG, H, KG, v_1, i_B, β, and f_k.

$$\theta = \alpha_1 e^{\alpha_1 t}(A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) + e^{\alpha_1 t}(-A_1 \beta_1 \sin \beta_1 t + A_2 \beta_1 \cos \beta_1 t)$$

$$+ \frac{2d_1 t}{c_1} \quad - \frac{2b_1 d}{c_1^2} \qquad (B55)$$

$$\ddot{\theta} = (\alpha_1^2 - \beta_1^2)\, e^{\alpha_1 t} (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t)$$

$$+ 2\, \alpha_1 \beta_1 e^{\alpha_1 t} (-A_1 \sin \beta_1 t + A_2 \cos \beta_1 t) + \frac{2\, d_1}{c_1}$$

Solution for z During Crushing

The summation of forces in the z-direction can be expressed in terms of z and t by the substitution of equation (B41).

The solution becomes

$$z = e^{\alpha_2 t}(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t) + \frac{d_2}{c_2} t^2 - \frac{2b_2 d_2}{c_2^2} - B_1 \qquad \text{(B62)}$$

where α_2, β_2, B_1, B_2, b_2, c_2, and d_2 are constants reflecting the influence of m_z, k_h, T_f (pounds per foot immersion, to be explained later), v_1, i_B, β, and f_k.

$$\dot{z} = \alpha_2 e^{\alpha_2 t}(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t) + e^{\alpha_2 t}(-B_1 \beta_2 \sin\beta_2 t + B_2 \beta_2 \cos \beta_2 t)$$

$$+ \frac{2\, d_2 t}{c_2} - \frac{2\, b_2 d_2}{c_2^2} \qquad \text{(B63)}$$

$$\ddot{z} = (\alpha_2^2 - \beta_2^2) e^{\alpha_2 t}(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t) \qquad \text{(B64)}$$

$$+ 2\alpha_2 \beta_2\, e^{\alpha_2 t}(-B_1 \sin \beta_2 t + B_2 \cos \beta_2 t) + \frac{2\, d_2}{c_2}$$

Mass and Mass Moment of Inertia

The values of m_x, m_z, and I_θ are needed to solve the previously expressed equations for x, z, and θ. These values of mass and mass moment of inertia (for movement as in pitching about the center of gravity) must be determined or suitably approximated based on the "given" parameters.

The underwater shape of polar icebreakers approximates that of a prolate ellipsoid as given by Saunders (24). Using a typical value for the ratio of L/B = 4.0, the following effective masses (body mass plus added mass) and mass moment of inertia are obtained:

$$m_x = 1.08 \frac{\Delta}{g} = 0.0336 \Delta \qquad \text{(B66)}$$

$$m_z = 1.86 \frac{\Delta}{g} = 0.0578 \Delta \qquad \text{(B67)}$$

$$I_\theta = 1.61 k^2 \frac{\Delta}{g} = 0.050 k^2 \Delta \qquad \text{(B68)}$$

where k = radius of gyration.

As indicated by Vosser (25), a reasonable value for radius of gyration for an icebreaker would be

$$k = 0.22 L \qquad \text{(B72)}$$

Substitution of that value in the equation for I θ gives a reasonable approximation.

Damping Coefficients

Values are needed for k_h (heave damping coefficient) and k_p (pitch damping coefficient) in order to solve the equations for x, z, and θ.

A good approximation which is relatively simple to use can be found published by Vosser (25). Based on his coefficients,

$$k_p = \frac{0.10}{g} \Delta L^{3/2} \quad \text{lb-ft-sec} \qquad (B69)$$

$$k_h = \frac{3.0}{g} \frac{\Delta}{L^{1/2}} \quad \text{lb-sec/ft} \qquad (B70)$$

Pounds per Foot Immersion

For sea water,

$$T_f = (64.2)\, L\, B\, \alpha \quad \text{for sea water} \qquad (B71)$$

Solution of Bow Forces During Crushing

In the development of the summation equations two important substitutions were made. These can be used to determine the components of the force at the bow.

$$F_{BXC} = k_1 x^2 \qquad (B34)$$

$$F_{BZC} = k_2 x^2 \qquad (B14)$$

where k_1 and k_2 reflect the influence of σ , i_B, β, and f_k.

Termination of the Crushing Phase, State 2.

Point A is defined as a point on the bow of the icebreaker at the waterline (point of force application from the ice). When this velocity has a direction which is forward and upward at an angle equal to the angle of the bow plus the trim there is no more penetration of the ice and crushing has ceased. See Figures B-XVIII and B-XIX.

Let $(GA)_x$ equal the horizontal distance from the center of gravity to the point of contact. Let $(GA)_z$ equal the vertical distance.

If γ is the angle from the horizontal to the velocity direction,

$$\tan \gamma = \frac{(GA)_x \dot{\theta} - \dot{z}}{\dot{x} - (GA)_z \dot{\theta}}$$

It may be seen that $\tan \gamma$ (or γ) is a function of t.

When

$$\tan \gamma = \tan (i_B + \theta)$$

crushing has ceased and State 2 is reached.

At this point sliding will commence (presuming the icebreaker still has forward velocity).

Sliding Phase, General

The sliding phase commences at State 2, when local crushing has ceased. The bow forces are no longer a function of the penetration. However, the vertical component and the horizontal component are interrelated.

During the sliding phase, the point of contact is assumed fixed (since further crushing would be negligible) and is assumed to be at the level of the waterline.

Bow Forces During Sliding

Figure B-XX illustrates the forces on the bow during sliding.

The force normal to the plating on each side is N/2. The friction force on each side is then $f_x \frac{N}{2}$.

In order to resolve these forces into components, Figure B-XXII, let

$$a_s = \cos\beta \sin i_B + f_k \cos i_B \tag{B80}$$

$$b_s = \cos\beta \cos i_B - f_k \sin i_B \tag{B81}$$

After linearizing and using trigonometric substitution

$$F_{BZS} = N(b_s - a_s\theta) \tag{B82}$$

$$F_{BXS} = N(\quad{}_s + b_s\theta) \tag{B83}$$

It follows that

$$F_{BXS} = F_{BZS}\left[\frac{a_s}{b_2} + \frac{a_2}{b_2}\left(\frac{a_s^2 + b_s^2}{a_s\, b_s}\right)\theta \right] \qquad \text{(B85)}$$

This allows substitution later on.

Icebreaking Thrust

Of the total thrust, part is being used to overcome the resistance of the water and the other part is used against the ice. This latter part, or the thrust produced in excess of the requirement to maintain a velocity in water is called the "icebreaking thrust".

At State 1,

$$T\,(1 - t) = R_T \qquad \text{(B21)}$$

where T is thrust, t is the thrust deduction factor, and R_T is total resistance (no ice at State 1).

Then "icebreaking thrust" may be expressed

$$T_{IB} = T(1 - t) - R_T$$

R_T may be broken down into residual and frictional resistance. After breaking it down and making suitable substitutions based on the assumption that the rotational propeller speed remains constant,

$$T_{IB} = T_{BOL} - K_5 v - \left[\frac{T_{BOL} - K_5 v_1}{v_1^2}\right] v^2 \qquad \text{(B27)}$$

where T_{BOL} is bollard thrust assuming r p m constant.

An illustrative plot of equation (B27) is shown in Figure B-XVI It is seen that State 1 there is no thrust being used against the ice. At the conclusion of sliding all thrust is being used against the ice and none to propel the ship through the water.

The term K_5 in equation (B27) is based on many parameters which would not be known at the early stages of design. It may be seen in Figure B-XVI that a linear approximation is suitable and in order.

$$T_{IB} = T_{BOL} \left(1 - \frac{v}{v_1}\right)$$

where $v = \frac{dx}{dt} = v_1$ = impact velocity.

Newton's Laws of Motion During Sliding

Figure B-XXIII illustrates the free body diagram for the sliding phase. Newton's laws of motion may be applied to the three types of motion encountered (x-direction, z-direction, and rotationally about the y-axis as in pitching). The three equations resulting are not independent.

In the forward direction,

$$\sum F_x = m_x \frac{d^2x}{dt^2}$$

$$T_{IB} \cos \theta - F_{BXS} = m_x \frac{d^2x}{dt^2} \qquad \text{(B89)}$$

This becomes,

$$T_{BOL} - \left(\frac{T_{BOL}}{v_1}\right) \dot{x} - \frac{a_s}{b_s} F_{BZS} - \frac{a_s}{b_s} \left(\frac{a_s^2 + b_s^2}{a_s b_s}\right) F_{BZS}\, \theta$$

$$- m_x \ddot{x} = 0 \qquad \text{(B90)}$$

In the downward vertical direction (z-direction),

$$\sum F_z = m_z \frac{d^2 z}{dt^2}$$

$$-F_{BZS} - T_{IB}\theta - \Delta - T_f h - k_h \frac{dz}{dt} + \Delta = m_z \frac{d^2 z}{dt^2} \qquad \text{(B91)}$$

This equation may be used to express F_{BZS} following substitution to eliminate T_{IB} and h.

$$F_{BZS} = - T_{BOL}\theta + \frac{T_{BOL}}{v_1} \dot{x}\, \theta - T_f z - T_f\, (LCG\text{-}LCF)\, \theta$$

$$-k_h \dot{z} - m_z \ddot{z} \qquad \text{(B92)}$$

In order to obtain an equation containing x, z, and θ as the only unknowns, equation (B92) is substituted into equation (B90) and then linearized. This equation is expressed as follows:

$$a_{11}\ddot{x} + b_{11}\dot{x} + c_{11}x + a_{12}\ddot{z} + b_{12}\dot{z} + c_{12}z + a_{13}\ddot{\theta} + b_{13}\dot{\theta} + c_{13}\theta = d_1 \qquad \text{(B95)}$$

where c_{11}, a_{13}, and b_{13} are zero. All other coefficients are constant and represent the influence of L, B, α, m_x, m_z, T_{BOL}, i_B, β, k_h, $\dot{x}_2$, $\dot{z}_2$, $\ddot{z}_2$, θ_2 and f_k .

The summation of moments (counterclockwise) is taken about the center of gravity.

$$\sum M_g = I_\theta \frac{d^2\theta}{dt^2}$$

$$F_{BZS} (GA)_x + F_{BXS} (GA)_z + T_{IB} \cos \theta \, (KG - d)$$

$$- (\Delta + T_f h) \, GM_L \theta - k_p \frac{d\theta}{dt} - I_\theta \frac{d^2\theta}{dt^2} = 0 \qquad \text{(B96)}$$

Substitution for F_{BZS}, F_{BXS}, $(GA)_x$, $(GA)_z$, T_{IB} and h leads to an equation containing x, z, and θ. This equation (B98), contains 43 terms and reflects the influence of all 16 parameters. Linearizing produces more terms but eventually the equation becomes,

$$a_{21}\ddot{x} + b_{21}\dot{x} + c_{21}x + a_{22}\ddot{z} + b_{22}\dot{z} + c_{22}z + a_{23}\ddot{\theta} + b_{23}\dot{\theta} + c_{23}\theta = d_2 \qquad \text{(B100)}$$

where $a_{21} = 0$. All other coefficients are constant and reflect the influence of all 16 parameters.

Newton's laws have allowed us to express two equations containing three unknowns, x, z, θ (and their derivatives).

Location Geometry

During the sliding phase the bow is in contact with a fixed point on the ice. Figure B-XXIV illustrates the relationship of $(\theta - \theta_2)$, $(z - z_2)$, and $(x - x_2)$.

It is seen that,

$$(x - x_2) = \frac{(GA)_{x2}(\theta - \theta_2) - (z - z_2)}{\tan\,(i_B + \theta)} + (GA)_{z2}\,(\theta - \theta_2)$$

where the subscript 2 indicates the initial condition for sliding, State 2.

Substitutions are made and then all non-linear terms are linearized and this leads to

$$a_{31}\ddot{x} + b_{31}\dot{x} + c_{31}x + a_{32}\ddot{z} + b_{32}\dot{z} + c_{32}z + a_{33}\ddot{\theta} + b_{33}\dot{\theta} + c_{33}\theta = d_3 \qquad \text{(B103)}$$

where c_{31}, c_{32}, c_{33}, and d_3 are constants reflecting the influence of i_B, $(GA)_{x2}$, $(GA)_{z2}$, θ_2, z_2, and x_2. Other coefficients are zero.

This gives us our necessary third equation.

Simultaneous Equations of Sliding

There are three equations of sliding to be solved simultaneously, equations (B95), (B100), and (B103) respectively. There are three unknowns, x, z, and θ (and their derivatives). These equations are the basis for the solution of sliding motion.

$$a_{11}\ddot{x} + b_{11}\dot{x} + c_{11}x + a_{12}\ddot{z} + b_{12}\dot{z} + c_{12}z + a_{13}\ddot{\theta} + b_{13}\dot{\theta} + c_{13}\theta = d_1$$

$$a_{21}\ddot{x} + b_{21}\dot{x} + c_{21}x + a_{22}\ddot{z} + b_{22}\dot{z} + c_{22}z + a_{23}\ddot{\theta} + b_{23}\dot{\theta} + c_{23}\theta = d_2$$

$$a_{31}\ddot{x} + b_{31}\dot{x} + c_{31}x + a_{32}\ddot{z} + b_{32}\dot{z} + c_{32}z + a_{33}\ddot{\theta} + b_{33}\dot{\theta} + c_{33}\theta = d_3$$

There must be an operation performed on these equations in order to solve them. The method chosen, since it incorporates initial conditions, is that of the LaPlace Transform.

The three equations become, as a result of the LaPlace transformation,

$$(a_{11}s^2 + b_{11}s)\ \mathcal{L}(x) + (a_{12}s^2 + b_{12}s + c_{12})\ \mathcal{L}(z) + (c_{13})\ \mathcal{L}(\theta) =$$

$$a_{11}s\ x_2 + a_{11}\dot{x}_2 + b_{11}x_2 + a_{12}s\ z_2 + a_{12}\dot{z}_2 + b_{12}z_2 + d_1/s$$

$$(b_{21}s + c_{21})\ \mathcal{L}(x) + (a_{22}s^2 + b_{22}s + c_{22})\ \mathcal{L}(z) + (a_{23}s^2 + b_{23}s + c_{23})\ \mathcal{L}(\theta) =$$

$$b_{21}x_2 + a_{22}s\, z_2 + a_{22}\dot{z}_2 + b_{22}z_2 + a_{23}\, s\, \theta_2 + a_{23}\dot{\theta}_2 + b_{23}\theta_2 + {d_2}/{s}$$

$$(c_{31})\, \mathcal{L}(x) + (c_{32})\, \mathcal{L}(z) + (c_{33})\, \mathcal{L}(\theta) = {d_3}/{s}$$

The right hand terms may be grouped into new constants by collecting coefficients of like powers of s. (For example $d_{11} = a_{11}\dot{x}_2 + b_{11}x_2 + a_{12}\dot{z}_2 + b_{12}z_2$).

The three simultrneous equations may now be written in shorter form.

$$(a_{11}s^2 + b_{11}s)\, \mathcal{L}(x) + (a_{12}s^2 + b_{12}s + c_{12})\, \mathcal{L}(z) + (c_{13})\, \mathcal{L}(\theta) =$$

$$d_{11} + d_{12}s + d_{13}/s$$

$$(b_{21}s + c_{21})\, \mathcal{L}(x) + (a_{22}s^2 + b_{22}s + c_{22})\, \mathcal{L}(z) + (a_{23}s^2 + b_{23}s + c_{23})\, \mathcal{L}(\theta) =$$

$$d_{21} + d_{22}s + d_{23}/s$$

$$(c_{31})\, \mathcal{L}(x) + (c_{32})\, \mathcal{L}(z) + (c_{33})\, \mathcal{L}(\theta) = d_{33/s}$$

Now we have three equations each containing the same three unknowns, $\mathcal{L}(x)$, $\mathcal{L}(z)$, and $\mathcal{L}(\theta)$.

Each of the unknowns may be resolved starting by using determinants.

$$\mathcal{L}(x) = \frac{\begin{vmatrix} (d_{11} + d_{12}s + d_{13}/s) & (a_{12}s^2 + b_{12}s + c_{12}) & (c_{13}) \\ (d_{21} + d_{22}s + d_{23}/s) & (a_{22}s^2 + b_{22}s + c_{22}) & (a_{23}s^2 + b_{23}s + c_{23}) \\ (d_{33}/s) & (c_{32}) & (c_{33}) \end{vmatrix}}{\begin{vmatrix} (a_{11}s^2 + b_{11}s) & (a_{12}s^2 + b_{12}s + c_{12}) & (c_{13}) \\ (b_{21}s + c_{21}) & (a_{22}s^2 + b_{22}s + c_{22}) & (a_{23}s^2 + b_{23}s + c_{23}) \\ (c_{31}) & (c_{32}) & (c_{33}) \end{vmatrix}}$$

The determinants for $\mathcal{L}(z)$ and $\mathcal{L}(\theta)$ are expressed in similar manner. Each has the same denominator.

After collecting coefficients of like powers of s, the $\mathcal{L}(x)$ numerator becomes,

$$N_{13}s^2 + N_{12}s^2 + N_{11}s + N_{10} + N_{09}/s \qquad \text{(B108)}$$

The numerator for the $\mathcal{L}(z)$ becomes,

$$N_{23}s^3 + N_{22}s^2 + N_{21}s + N_{20} + N_{19}/s \qquad \text{(B152)}$$

The numerator for the $\mathcal{L}(\theta)$ becomes,

$$N_{33}s^3 + N_{32}s^2 + N_{31}s + N_{30} + N_{29/s} \qquad \text{(B166)}$$

The denominator, which is common to all three, becomes,

$$D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0 \qquad \text{(B109)}$$

Therefore,

$$\mathcal{L}(x) = \frac{N_{13}s^4 + N_{12}s^3 + N_{11}s^2 + N_{10}s + N_{09}}{s\left[D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0\right]} \qquad \text{(B110)}$$

The $\mathcal{L}(z)$ and $\mathcal{L}(\theta)$ may be expressed in similar fashion. It is noted that this may be expressed as a proper fraction; the denominator is one power higher than the numerator.

By letting $a_{4x} = {}^{N}13/D_4$, $a_{3x} = {}^{N}12/D_4$, $a_{x2} = {}^{N}11/D_4$,

$a_{1x} = {}^{N}10/D_4$, and $a_{0x} = {}^{N}09/D_4$

along with

$b_4 = {}^{D}3/D_4$, $b_3 = {}^{D}2/D_4$, $b_2 = {}^{D}1/D_4$,

and $b_1 = {}^{D}0/D_4$,

$$\mathcal{L}(x) = \frac{a_{4x}s^4 + a_{3x}s^3 + a_{2x}s^2 + a_{1x}s + a_{0x}}{s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s} \qquad \text{(B113)}$$

The numerator may be written as

$$(s - s_1)\,(s^4 + b_4 s^3 + b_3 s^2 + b_2 s + b_1)$$

where $s_1 = 0$.

Biquadratic Solution

The biquadratic appearing in the denominator was presumed to have two pairs of complex conjugate roots. (This was after many other attempts proved unmeaningful. Furthermore this would lead to damped oscillatory motion which would appear typical in the physical case if the icebreaker were to slide back if unrestricted by static friction.)

This means that

$$(s^4 + b_4 s^3 + b_3 s^2 + b_2 s + b_1) =$$

$$\left[(s + \alpha_3)^2 + \beta_3^2\right]\left[(s + \alpha_4)^2 + \beta_4^2\right] \qquad \text{(B115)}$$

Let $\alpha_3 = B_1$, $\beta_3 = A_1$, $\alpha_4 = B_2$, and $\beta_4 = A_2$

Then the right hand side becomes

$$\left[(s_4 B_1)^2 + A_1^2\right]\left[(s+B_2)^2 + A_2^2\right] = \left[s^2 + 2B_1 s + (A_1^2 + B_1^2)\right]\left[s^2 + 2B_2 s + (A_2^2 + B_2^2)\right]$$

Equation (E115) may be written as

$$\frac{s^4 + b_4 s^3 + b_3 s^2 + b_2 s + b_1}{s^2 + 2\,B_1 s + (A_1^2 + B_1^2)} = s^2 + 2B_2 s + (A_2^2 + B_2^2)$$

If the division is carried out on the left, that side becomes

$$s^2 + (b_4 - 2B_1)s + \left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right]$$

and this must equal

$$s^2 + 2B_2 s + (A_2^2 + B_2^2)$$

Therefore,

$$2B_2 = (b_4 - 2B_1) \qquad \text{(E117)}$$

$$A_2^2 + B_2^2 = \left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right]$$

However, the division carried out above has a remainder, and this remainder must be set equal to zero.

$$\left[b_2 - (b_4 - 2B_1)(A_1^2 + B_1^2)\right] - \left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right]\left[2B_1\right] = 0 \qquad \text{(E118)}$$

and

$$b_1 - \left[b_3 - A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right]\left[A_1^2 + B_1^2\right] = 0 \qquad \text{(E120)}$$

Now there are two equations with two unknowns, A_1 and B_1. The unknowns are real numbers.

The solution of these two equations for B_1 leads to an equation of the form

$$a_6 B_1^6 + a_5 B_1^5 + a_4 B_1^4 + a_3 B_1^3 + a_2 B_1^2 + a_1 B_1 + a_0 = 0$$

where all coefficients are known constants reflecting the influence of b_4, b_3, b_2, and b_1.

It is noted that we started to find the roots of a fourth order polynomial and now it is "simplified" to a sixth order. Actually this is simpler; the unknown, B_1, is a real number. (The roots of the quartic are not.)

It is discovered in the complete development that

$$0 < B_1 < \frac{b_4}{4} \qquad \text{or} \qquad \frac{b_4}{4} < B_1 < \frac{b_4}{2} \tag{B121}$$

Therefore, B_1 is best solved by a trial and error iterative solution starting with B_1 near zero.

Once B_1 has been determined, the other values may be determined.

$$\alpha_3 = B_1 \tag{B124}$$

$$\beta_3 = \sqrt{\frac{-b_2 - 3b_4\alpha_3^2 + 4\alpha_3^3 + 2b_3\alpha^3}{4\,\alpha^3 - b_4}} \tag{B125}$$

$$\alpha_4 = \frac{1}{2}(b_4 - 2\alpha_3) \tag{B126}$$

$$\beta_4 = \sqrt{b_3 - (\beta_3^2 + \alpha_3^2) - 4\alpha_3\alpha_4 - \alpha_4^2} \qquad \text{(B127)}$$

We now have the solution to the denominator of equation (B113).

Partial Fraction Form

It is necessary to put equation (B113) into the form of partial fractions in order to take the inverse LaPlace.

$$\mathcal{L}(x) = \frac{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}{s\left[(s+\alpha_3)^2 + \beta_3^2\right]\left[(s+\alpha_4)^2 + \beta_4^2\right]} =$$

$$\frac{A_1}{s} + \frac{A_3 s + B_3}{\left[(s+\alpha_3)^2 + \beta_3^2\right]} + \frac{A_4 s + B_4}{\left[(s+\alpha_4)^2 + \beta_4^2\right]}$$

The right hand side contains five unknowns. The right hand term is put into the form of a polynomial with a common denominator. The coefficients of like terms in the numerator are collected and set equal to the equivalent coefficient of a like term in the numerator of the left side. For example,

$$a_4 s^4 = (\text{coefficients on right of } s^4)\, s^4$$

$$a_4 = (\text{coefficients of } s^4 \text{ from right side})$$

$$a_4 = A_1 + A_3 + A_4 \qquad \text{(B128)}$$

$$a_3 = 2\alpha_4 A_1 + 2\alpha_3 A_1 + 2\alpha_4 A_3 + B_3 + 2\alpha_3 A_4 + B_4 \qquad \text{(B129)}$$

$$a_2 = - - - - - - - - - - - - - \qquad (B130)$$

$$a_1 = - - - - - - - - - - - - - \qquad (B131)$$

$$a_0 = (\alpha_3^2 + \beta_3^2)(\alpha_4^2 + \beta_4^2)\ A_1 \qquad (B132)$$

These five equations contain five unknowns. They may be reduced to four equations with four unknowns by the substitution of A_1 (from equation (B132)) into all other equations.

After further substitution involving d_1, d_2, d_3, and d_4 as previously defined, the four equations may be solved by the use of a matrix.

$$\begin{array}{ccccc}
(A_3) & (A_4) & (B_3) & (B_4) & \\
\left[\begin{array}{c} 1 \\ 2\alpha_4 \\ g_4 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 2\alpha_3 \\ g_3 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2\alpha_4 \\ g_4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2\alpha_3 \\ g_3 \end{array} & \left.\begin{array}{c} d_1 \\ d_2 \\ d_3 \\ d_4 \end{array}\right]
\end{array}$$

where $g_3 = (\alpha_3^2 + \beta_3^2)$

$g_4 = (\alpha_4^2 + \beta_4^2)$

The solution of the matrix leads to

$$B_4 = \frac{2(\alpha_3-\alpha_4)\left[g_4d_3-g_4^2d_1-2\alpha_4d_4\right] + (g_3-g_4)\left[-g_4d_2+2\alpha_4g_4d_1 + d_4\right]}{2(\alpha_3-\alpha_4)\left[2\alpha_3g_4-2\alpha_4g_3\right] + (g_3-g_4)^2}$$

$$B_3 = \frac{d_4 - g_3B_4}{g_4} \qquad \text{(B142)}$$

$$A_4 = \frac{g_4(d_2-2\alpha_4d_1) - d_4 + B_4\,(g_3-g_4)}{2g_4\,(\alpha_3 - \alpha_4)} \qquad \text{(B143)}$$

$$A_3 = d_1 - A_4 \qquad \text{(B144)}$$

One more substitution is in order.

$$\phi_{23} = \beta_3A_3 \qquad \phi_{24} = \beta_4A_4 \qquad \text{(B146)}$$

$$\phi_{13} = B_3 - \alpha_3A_3 \qquad \phi_{14} = B_4 - \alpha_4A_4$$

Now we have $\mathcal{L}(x)$ with all terms known and in useable form.

$$\mathcal{L}(x) = \frac{A_1}{s} + \frac{(s+\alpha_3)\phi_{23}+\beta_3\phi_{13}}{\beta_3\left[(s+\alpha_3)^2+\beta_3^2\right]} + \frac{(s+\alpha_4)\phi_{24} + \beta_4\phi_{14}}{\beta_4\left[(s+\alpha_4)^2 + \beta_4^2\right]} \qquad \text{(B145)}$$

$$x = \mathcal{L}^{-1}(x) = A_1 + \frac{1}{\beta_3} e^{-\alpha_3 t} (\phi_{23} \cos \beta_3 t + \phi_{13} \sin \beta_3 t)$$

$$+ \frac{1}{\beta_4} e^{-\alpha_4 t} (\phi_{24} \cos \beta_4 t + \phi_{14} \sin \beta_4 t) \qquad \text{(B147)}$$

Differentiating with respect to time, t.

$$\dot{x} = \frac{-\alpha_3}{\beta_3} e^{-\alpha_3 t} (\phi_{23} \cos \beta_3 t + \phi_{13} \sin \beta_3 t)$$

$$+ e^{-\alpha_3 t} (-\phi_{23} \sin \beta_3 t + \phi_{13} \cos \beta_3 t)$$

$$\frac{-\alpha_4}{\beta_4} e^{-\alpha_4 t} (\phi_{24} \cos \beta_4 t + \phi_{14} \sin \beta_4 t)$$

$$+ e^{-\alpha_4 t} (-\phi_{24} \sin \beta_4 t + \phi_{14} \cos \beta_4 t) \qquad \text{(B150)}$$

$$\ddot{x} = \frac{(\alpha_3^2 - \beta_3^2)}{\beta_3} e^{-\alpha_3 t} (\phi_{23} \cos \beta_3 t + \phi_{13} \sin \beta_3 t)$$

$$-2\alpha_3 e^{-\alpha_3 t} (-\phi_{23} \sin \beta_2 t + \phi_{13} \cos \beta_3 t)$$

$$+ \frac{(\alpha_4^2 - \beta_4^2)}{\beta_4} e^{-\alpha_4 t} (\phi_{24} \cos \beta_4 t + \phi_{14} \sin \beta_4 t)$$

$$- 2\alpha_4 e^{-\alpha_4 t} (-\phi_{24} \sin \beta_4 t + \phi_{14} \cos \beta_4 t) \qquad \text{(B151)}$$

The equations given above represent the complete sliding motion of the center of gravity in the x-direction. The coefficients used are shown somewhat generalized here in that they do not carry the subscript x. For example,

$$\phi_{23} = P_{23x}$$

$$d_1 = d_{1x}$$

$$a_4 = a_{4x} \quad \text{etc.}$$

In order to solve for z and θ (along with their derivatives) it is only necessary to recognize that

$$\mathcal{L}(z) = \frac{N_{23}s^4 + N_{22}s^3 + N_{21}s^2 + N_{20}s + N_{19}}{s\left[D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0\right]} \tag{B153}$$

and

$$\mathcal{L}(\theta) = \frac{N_{33}s^4 + N_{32}s^3 + N_{31}s^2 + N_{30}s + N_{29}}{s\left[D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0\right]} \tag{B167}$$

Except for the values of the constant coefficients in the respective numerators, these equations are identical to equation (B110). The method of solution for z and θ is identical to that of the solution for x. The resulting equations are identical except for subscript. For example,

$$z = c_{1z} + \frac{1}{\beta_3} e^{-\alpha_3 t} (P_{23z}\cos\beta_3 t + P_{13z}\sin\beta_3 t)$$

$$+ \frac{1}{\beta_4} e^{-\alpha_4 t} (P_{24z}\cos\beta_4 t + P_{14z}\sin\beta_4 t) \qquad \text{(B163)}$$

$\dot{z}$ and $\ddot{z}$ are obtained by differentiation and are given by equations (B164) and (B165).

$$\theta = c_{1\theta} + \frac{1}{\beta_3} e^{-\alpha_3 t} (P_{23\theta}\cos\beta_3 t + P_{13\theta}\sin\beta_3 t)$$

$$+ \frac{1}{\beta_4} e^{-\alpha_4 t} (P_{24\theta}\cos\beta_4 t + P_{14\theta}\sin\beta_4 t) \qquad \text{(B177)}$$

$\dot{\theta}$ and $\ddot{\theta}$ are obtained by differentiation and are given by equations (B68) and (B69).

The equations completely describe the motion of the icebreaker during the sliding phase.

Vertical Force on Bow During Sliding

In the previous solution, F_{BZS}, the vertical bow force during sliding, was eliminated by substitution. Equation (B92) gives the substitution. Equation (B92) gives the value of F_{BZS} directly.

$$F_{BZS} = - T_{BOL}\theta + \frac{T_{BOL}}{V_1} \dot{x}\,\theta - T_f\, z$$

$$- T_f\,(LCG-LCF)\,\theta - k_h\dot{z} - m_z\ddot{z}$$

where θ, $\dot{x}$, z, $\dot{z}$, and $\ddot{z}$ are obtained from equations (B67), (B150), (B163), (B164), and (B165) respectively.

Termination of Sliding Phase, State 3

The equations of velocity, (B150), (B164), and (B178), may be combined vectorially to indicate the velocity of a point on the bow in contact with the ice. When this velocity (or likewise the horizontal component) becomes zero the sliding between ship and ice has terminated; State 3 is reached.

The z-component of the bow velocity is

$$v_{az} = \dot{z} - (GA)_x \dot{\theta} \qquad \text{(B161)}$$

The x-component is

$$v_{ax} = \dot{x} - (GA)_z \dot{\theta} \qquad \text{(B162)}$$

When the velocity of the bow relative to the ice becomes zero each of the components becomes zero. It is therefore sufficient to use either one to define State 3.

$$v_{ax} = \dot{x} - \left[(H-KG) + z \right] \dot{\theta} \qquad \text{(B183)}$$

For each value of t (time, during sliding) there is a value of v_{ax}. When, by iteration, $v_{ax} = 0$, that time is assigned the symbol t_3

and all other values can be determined using t_3.

Static Equilibrium, State 4

Presuming the icebreaker does not immediately back off, a slight amount of settling is going to take place with the bow remaining held in the ice at a position defined by State 3. If all velocities and accelerations were zero at State 3 there would be no "settling". However, this seems rather unlikely. For that reason, the static equilibrium problem will be solved using a point of support as defined by State 3. See Figure B-XXVII (Thrust has been dropped from the equilibrium solution. The screws must be stopped at some point anyway and this will lead to the higher value of sustained downward force under the bow.)

$$\sum F_x = 0 \qquad \text{(B184)}$$

$$\sum F_z = -F_{BZ4} - (\Delta + T_f h_4) + \Delta = 0 \qquad \text{(B185)}$$

$$\sum M_G = F_{BZ4}(GA_4)_x - (\Delta + T_f h_4) GM_L \theta_4 = 0 \qquad \text{(B187)}$$

where $h_4 = z_3 + (LCG\text{-}LCF)\theta_4 + (GA_3)_x(\theta_4 - \theta_3)$ (B186)

Equation (B186) can be substituted into equations (B185) and (B187). This gives us two equations with two unknowns, θ_4 and F_{BZ4}.

Combining those two equations leads to one equation,

$$a_4 F_{BZ4}^2 + b_4 F_{BZ4} + c_4 = 0 \qquad (B194)$$

where a_4, b_4, and c_4 are constants reflecting the influence of $(GA_3)_x$, θ_3, z_3 and the hydrostatic properties of the icebreaker.

The meaningful solution of equation (B194) is

$$F_{BZ4} = \frac{-b_4 + \sqrt{b_4^2 - 4a_4 c_4}}{2\ a_4} \qquad (B195)$$

This value is the object of this research.

Incidentally, the final position, which may be of interest, may be readily determined.

The change in position (from State 3 to State 4) in the x-direction is negligible. The final trim, θ_4, may be obtained from equation (B193),

$$\theta_4 = \frac{-F_{BZ4}}{d_1 T_f} + \frac{z_3}{d_1} + \frac{(GA_3)_x\ \theta_3}{d_1} \qquad (B193)$$

where $d_1 = (LCG-LCF) + (GA_3)_x$

The final position of the center of gravity may be obtained from equation (B186),

$$z_4 = z_3 + (GA_3)_x\ (\theta_4 - \theta_3) \qquad (B186)$$

Extracting Thrust

Normally the static friction acting on the bow, once motion has stopped, reaches magnitudes greater than that of kinetic friction. It is possible, in fact probable, that the icebreaker may not slide back of its own accord. In that case backing thrust is necessary.

Actually any movement of the ship relative to the ice may free this static grip. This is where shifting the rudder, using heeling and/or trimming tanks, or setting off jarring blasts on the ice may help.

Most important though is the extracting thrust requirement, based on backing thrust sufficient in itself to free the icebreaker from this static grip.

Figure B-XXIX shows the forces acting on the bow. Figure B-XXX illustrates the free body diagram.

Solution of the free body diagram leads to the required extraction thrust,

$$E_t = \frac{F_{BZ4}}{(\frac{a_7}{b_7}) \cos \theta_4 - \sin \theta_4)} \qquad \text{(B202)}$$

where $a_7 = (\cos \beta) \cos (i_B + \theta_4) + f_s \sin (i_B + \theta_4)$

and $b_7 = - (\cos \beta) \sin (i_B + \theta_4) + f_s \cos (i_B + \theta_4)$

Computer Program

The solution of all previous equations is extremely lengthy and there are several iterative processes involved. The digital computer has made it feasible to solve the entire problem. In fact it has made comparisons and further study possible.

The following is a listing of the input data which must be supplied:

BP	Length between perpendiculars, ft.
B	Beam at waterline, ft.
H	Mean draft, ft.
DIS	Displacement, lbs.
BA	Bow angle (from base line to stem), radians
SA	Spread angle complement (normal to bow plating with respect to centerline plane), radians
V1	Impact velocity, ft/sec.
AL	α, Waterplane coefficient, dimensionless
CF	LCF, Longitudinal position of the center of flotation (- if aft of amidships, + if forward), ft.
CG	LCG, Longitudinal position of the center of gravity (- if aft of amidships, + if forward), ft.
GK	KG, Height of center of gravity above base line, ft.
D	Height of thrust line above base line near center of gravity, ft.
TB	Bollard thrust which would be obtained for rpm used during crushing and sliding, lbs.
GM	GM_L, Longitudinal metacentric height, ft.
FK	Ice/ship kinetic coefficient of friction, dimensionless

SIG Compressive failure stress of ice, lb/ft^2.

The most important output of the program is the relatively sustained downward force under the bow during State 4.

F_{BZ4} = Vertical Force at Bow, lbs.

In addition other output is available as follows:

X4 = Forward motion from initial point of contact, ft.

Z4 = Vertical position of the center of gravity relative to the original position at the time of contact, ft.

TH4 = θ_4, Final trim, radians

WRAT = "White Ratio" = $\dfrac{F_{BZ4}}{\text{(Displacement)(Impact velocity)}}$,sec/ft

ET = Extracting thrust, lbs.

RAT = Extracting thrust/Bollard thrust, dimensionless.

Other information is readily available (if desired) as a function of time.

X, XD, XDD = $x, \dot{x}, \ddot{x}$ Forward position and its derivatives (ft, ft/sec, and ft/sec^2)

Z, ZD, ZDD = $z, \dot{z}, \ddot{z}$ Vertical position of the center of gravity and its derivatives (ft, ft/sec, and ft/sec^2)

TH, THD, THDD = $\theta, \dot{\theta}, \ddot{\theta}$ Pitch angle and its derivatives (radians, rad/sec, and rad/sec^2)

F_{BZ} Downward force under bow during all phases as a function of time. lbs.

Other output is available directly but is only incidental to the solution of the basic problem. This includes total mass, including virtual (in each sense, x, z, θ), radius of gyration, pounds per foot immersion, pitch damping coefficient, heave damping coefficient, and scores of coefficients used in the solution.

Suitable Simplifications

Some of the information required for the solution may not be known with much accuracy during the design stage. For that reason suitable approximations are in order.

For example

$$BM_L = \frac{C_{i\ell} L^2}{C_b H} \tag{B206}$$

and $$KB = \frac{\alpha}{C_b + \alpha} H \tag{B207}$$

where $$C_{i\ell} = 0.030 + 0.1304 (\alpha - 0.65) \tag{B203}$$

C_b = Block coefficient

L = Length between perpendiculars

H = Draft

α = Waterplane coefficient

Other such approximations include KG (and therefore GM_L), and bollard thrust.

Parametric Study

The variation of a parameter certainly has an effect on the sustained downward force. There are sixteen input variables. (The static coefficient of friction is only for the determination of extraction thrust.)

Of the sixteen, the following may not be considered independent:

GM_L, LCF, α, Δ, H, B, LEP

A change in any one of these involves a change in another.

Some of the parameters are relatively independent within reasonable limits. These are as follows:

Bollard thrust, D (Height of thrust line), KG, LCG, β (the spread angle complement of the bow plating), and, perhaps most significant i_B (the bow angle relative to the base line.)

A few of the parameters may be considered completely independent. They are as follows:

v_1 (impact velocity), f_k (kinetic coefficient of friction), and (compressive failure stress of ice).

The "independent" variables will be varied over a suitable range to determine the effect on the downward force at the bow. The impact velocity will be varied along with each one. The remaining parameters will be assigned values representing the "Wind" Class Icebreaker. (Actually, the "Glacier" Class and the "Lenin" Class will be used also but the illustrations of result will be based on the "Wind" Class. Conclusions, unless noted to the contrary, will be valid for all three classes.)

"White Ratio"

For lack of a better name, the ratio is defined as

$$\text{WRAT} = \frac{F_{BZ4}}{(\text{DIS})(\text{V1})} \text{ sec/ft} \qquad \text{(B214)}$$

It is anticipated that the downward force under the bow will be affected approximately linearly with displacement and impact velocity. The coefficient (White Ratio) may be of use for approximate comparison of the parameter effects.

The "dependent variables" (GM_L, LCF, α, Δ, H, B, LBP) may be varied only by varying other parameters simultaneously. For example, a change in α, the waterplane coefficient, will cause a change in pounds-per-foot-immersion (accounted for automatically in the program), the height of the center of buoyance (KB), and the distance from the center of buoyance to the longitudinal metacenter (BM_L). Keeping displacement length, draft, and beam constant, the resulting change may be examined.

The longitudinal position of the center of flotation may be changed slightly and a change in form would then be necessary to keep displacement, length, draft, and beam constant. This shift is incorporated to find the effect.

The beam-to-draft ratio is varied to investigate the effect. (Displacement is held constant as is length.) A new solution for GM_L is necessary.

The length-to-beam ratio (frequently 4.0 in polar icebreakers) is varied to investigate the effect. (Displacement and draft are held constant.) A new solution for GM_L is necessary.

Displacement effect is investigated three ways. One is simply the comparison of three different classes of icebreakers (Wind, Glacier, Lenin).

A second way is by holding, for a given class, the length, draft, and beam constant while varying the block coefficient (and consequently displacement).

The third displacement comparison is to vary the size of a given class of icebreaker such that geometrically similar ships (geosims) are generated. For example, all lengths are multiplied by the scale factor; volumes(i.e. displacement) are multiplied by the scale factor cubed.

By means of the variations indicated above it is possible to determine what values (i.e. high, low) would lead to the generation of the maximum sustained downward generated by ramming.

Model Parameters

In order to model test it is necessary to multiply all ship linear dimensions by 1/2. See equations (B222) and (B223). Coefficients are dimensionless and are not changed. Likewise, bow angle and spread angle are not changed.

The ship displacement and the bollard thrust must be multiplied by $1/\lambda^3$. See equations (B226).

The compressive failure stress of the ice must be multiplied by $1/\lambda$. This, of course, implies that a different bow supporting medium must be used in model tests. (Care must be taken to adjust the coefficient of kinetic friction if necessary.)

Since gravity and dynamics are involved, it is necessary that ship and model be at equivalent Froude Numbers.

$$Vl_m = Vl_s/\sqrt{\lambda} \qquad \text{(B229)}$$

By using the above-mentioned scales, the scaled final position (State 4.) of model and ship will be identical and the downward forces under the bow will be related as follows:

$$F_m = F_s/\lambda^3 \qquad \text{(B230)}$$

The relationship of time of events for the ship compared to the time of events for the model is

$$\frac{T_S}{T_M} = \sqrt{\lambda} \qquad \text{(B231)}$$

III RESULTS

Prediction of Westwind Behavior

The problem of a "Wind Class" icebreaker ramming virtually unyielding ice is solved using the solution indicated in Chapter II. The parameters used are given in Table IV. Note that three different impact velocities are used 11.32 ft/sec (6.7 knots), 13.51 ft/sec,(8.0 knots), and 15.52 ft/sec (9.2 knots).

The solution to the problem includes x, z, and θ (as well as their respective first and second derivatives as functions of time. In addition, the downward force under the bow is determined as a function of time.

The choice of parameters is based on the characteristics of the Westwind at the time of tests run during the summer of 1963. During the period of contact with the ice, full throttle was used so the maximum value of bollard thrust is used in the program.

Figures VIII, IX and X are plots of the prediction of x, $\dot{x}$, $\ddot{x}$; z, $\dot{z}$, $\ddot{z}$; and θ, $\dot{\theta}$, $\ddot{\theta}$ as functions of time for the run (37B) with an impact velocity of 13.51 ft/sec.

Figure XI is a plot of the predictions of $\ddot{z}$, $\ddot{\theta}$, $\dot{\theta}$, and $\dot{x}$ as functions of time for an impact velocity of 11.32 ft/sec (Run 36B). Figure XIII is similar but for Run 37B and Figure XV is for Run 38B.

Figure XII is a plot of the prediction of the downward force under the bow as a function of time for Run 36B. (Figure XIV is for Run 37B

and Figure XVI is for Run 38B.)

It must be recalled that the solution is based on an icebreaker identical to the Westwind except that it was assumed the stem was straight and continuous from the waterline to the keel.

Observation of Westwind Trials

In the summer of 1963 trials were run using the C.G.C. Westwind off the northwest coast of Greenland. (37)

The following values pertain to the trial runs of interest (26):

BP = 250.0 ft

B = 64.0 ft

Draft frd 25.0 ft

Draft aft 27.5 ft

Mean draft H = 26.25'

BA (bow angle) = 0.523

SA (β) = 0.886

AL (α) = 0.724

SIG (failure stress of ice) = 19.6 kg/cm^2 (27)
in tension

Incidentally, the ice thickness exceeded 580 cm, or 19.0 ft.

By conversion,

$$\text{SIG} = 279.0 \text{ lb/in}^2 = 40{,}200 \text{ lb/ft}^2 \text{ (in tension)}$$

From the "Displacement and Other Curves" for the Westwind (28),

Trim = 2.5 ft by the stern
H = 26.25 ft
DIS = 5600 tons = 12,530,000.0 lb

From a "Wind Class" inclining experiment (29), for normal load,

GK (height of c.g.) = 23.4 ft
CF (longitudinal position of c.g. of waterplane) = -1.3 ft

Uncorrected LCB = - 2.4 ft

Moment to change trim 1" = 18.6 x 20 = 372 ft-tons

Moment = (372)(30) = 11,160 ft-tons

Shift of LCB aft = $\frac{11,160}{5,600}$ = 1.99 ft aft

LCG = LCG = -2.40 - 1.99 = -4.39 ft

Bow angle is increased due to trim by $\frac{2.5}{250.0}$ = 0.010 rad.

BA = 0.533

At the center of gravity, the thrust line is approximately 16 feet above the keel. Therefore,

D = 16.0

From equation (B205)

$$C_b = \frac{DIS}{64.2\ LBH} = \frac{12{,}530{,}000}{(64.2)(250)(64)(26.25)}$$

$$C_b = 0.464$$

From equation (B207)

$$KB = \frac{\alpha}{C_b + \alpha}\ H = \frac{(0.724)(26.25)}{(0.464 + 0.724)}$$

$$KB = \frac{(0.724)(26.25)}{(1.188)} = 16.0 \text{ ft.}$$

From equation (B203),

$$C_{i\ell} = 0.030 + 0.1304\ (\alpha - 0.65)$$

$$C_{i\ell} = 0.030 + 0.1304\ (0.724 - 0.650)$$

$$C_{i\ell} = 0.030 + 0.1304\ (0.074) = 0.0396$$

From equation (B206)

$$BM_L = \frac{C_{i\ell}\ L^2}{C_b\ H} = \frac{(0.0396)(250)^2}{(0.464)(26.25)} = 203 \text{ ft}$$

From equation (B211),

$$GM_L = KB + BM_L - KG$$

$$GM_L = (16.0) + (203.0) - (23.4) = 195.6 \text{ ft.}$$

Maximum thrust was used during the sliding and crushing phase. Maximum bollard thrust for the "Wind Class" is 270,000 lbs. (4).

In summary, the values given in Table IV pertain to the three trial runs of interest.

The observed behavior of these three runs is given in Figures XI, XIII and XV. These are plots of $\ddot{z}$, $\ddot{\theta}$ (measured by accelerometers) (the one for $\ddot{z}$ was mounted near the center of gravity.), θ (measured by gyro), and $\dot{x}$ (as measured using "Raydist". The value of $\dot{x}$ was not considered to be reliable according to DTMB personnel.

It is noted that the protrusion of the housing for the formerly installed bow propeller would come in contact with the ice after about 1.4 seconds in Run 36B (about 1.2 seconds for 37B.; about 1.0 seconds for 38B). For that reason, observed results are not plotted much beyond those times.

Figures XII, XIV and XVI are plots of the strain reading in the transverse direction at the lower portion of a forward transverse bulkhead. There is no direct correlation to the magnitude of the load at the bow. However, the strain on that bulkhead is primarily created by the bow load. For that reason it is plotted to show that the maximum peak load occurs about half a second after initial contact rather than when the icebreaker has come to a stop with its bow well up on the ice.

TABLE IV

PARAMETERS USED FOR FULL SCALE TEST COMPARISON

WESTWIND

BP = 250.0 ft

H = 26.25 ft

BA = 0.533 rad*

AL = 0.724

CG = -4.39 ft

D = 16.0

GM_L = 195.6 ft

FS = 0.8

B = 64.0 ft

DIS = 12,530,000 lbs

SA = 0.886

CF = -1.30 ft

GK = 23.4 ft

TB = 270,000 lbs

FK = 0.2

SIG = 144,000 lbs. (30)

Run 36B Impact velocity, V1 = 11.32 ft/sec

Run 37B Impact velocity, V1 = 13.51 ft/sec

Run 38B Impact velocity, V1 = 15.52 ft/sec

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* Slightly greater than 30° to account for initial trim.

Figure VIII

x, $\dot{x}$, and $\ddot{x}$ (Predicted)

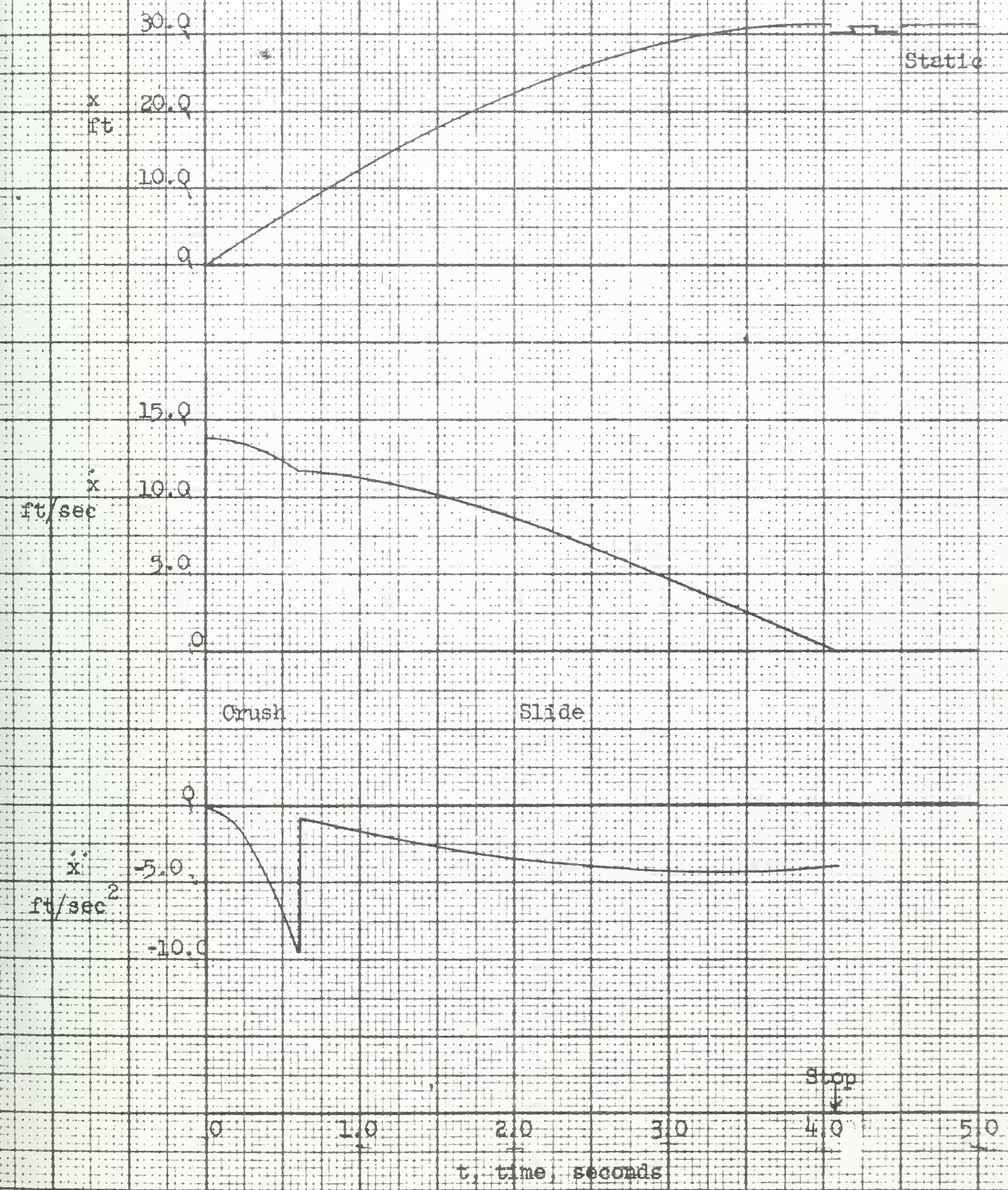
vs

TIME

Run 37B (Parameters given in Table IV)

R m W April 1965

RmW



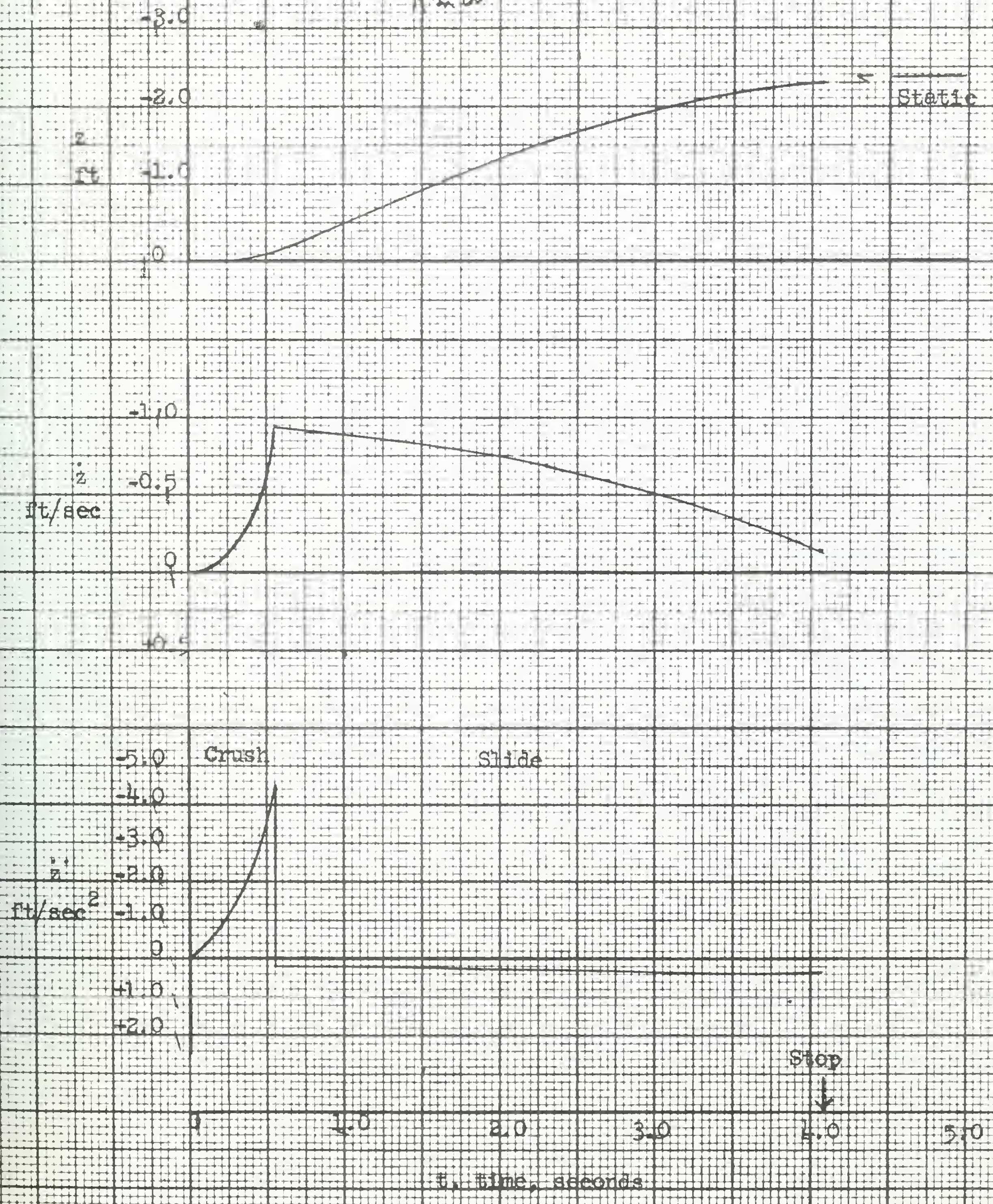


Figure IX
z, $\dot{z}$, and $\ddot{z}$ (Predicted)
vs
TIME
Run 37 B (Parameters given in Table IV)
R m W April 1965
RmW

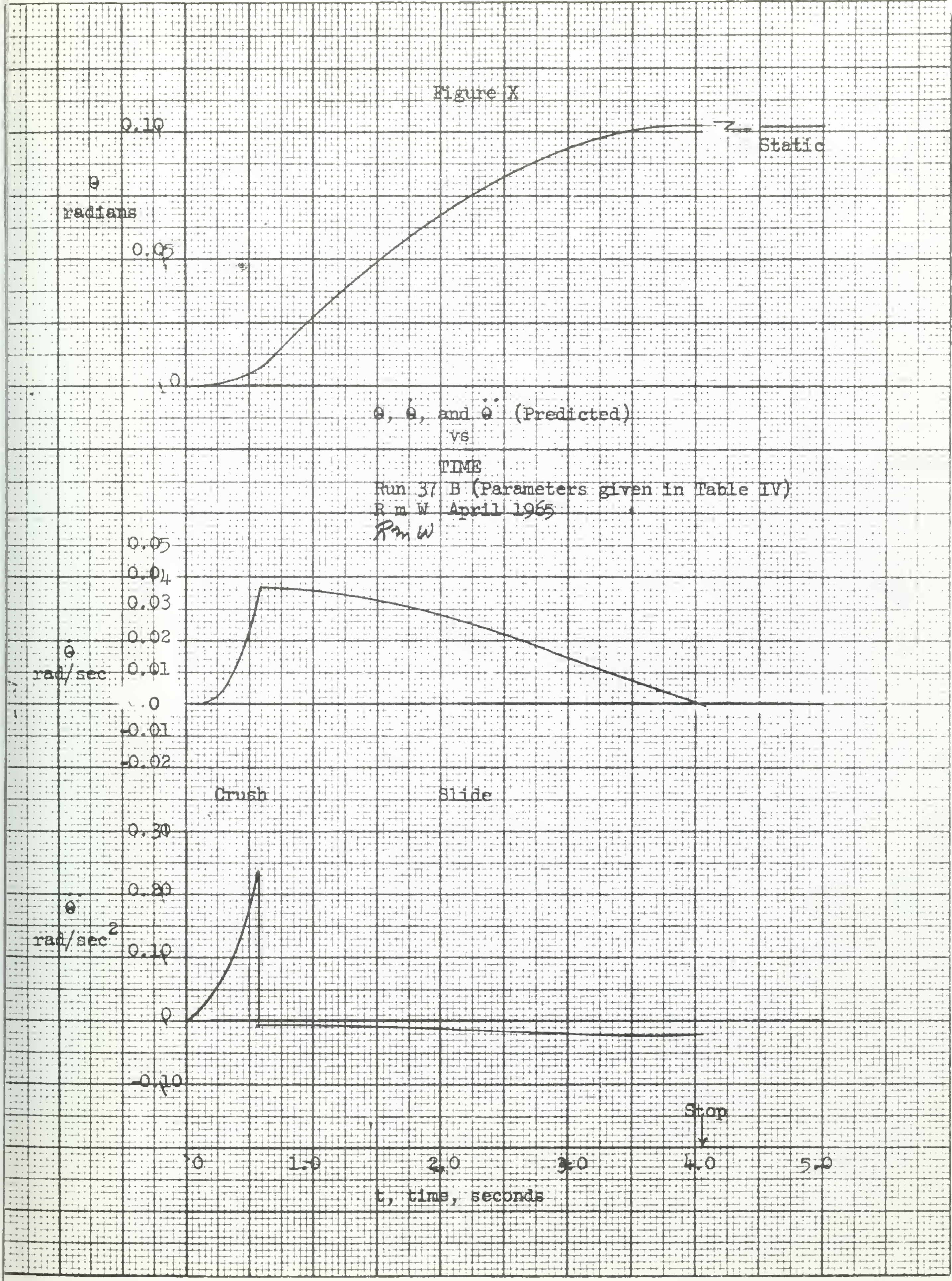


Figure X

θ, $\dot{\theta}$, and $\ddot{\theta}$ (Predicted)
vs
TIME
Run 37 B (Parameters given in Table IV)
R m W April 1965
RmW

Figure XI

$\ddot{z}$, θ, $\ddot{\theta}$, and x (Predicted and Observed)
vs
TIME

Run 36 B (Parameters given in Table IV)
R m W April 1965
Rmw

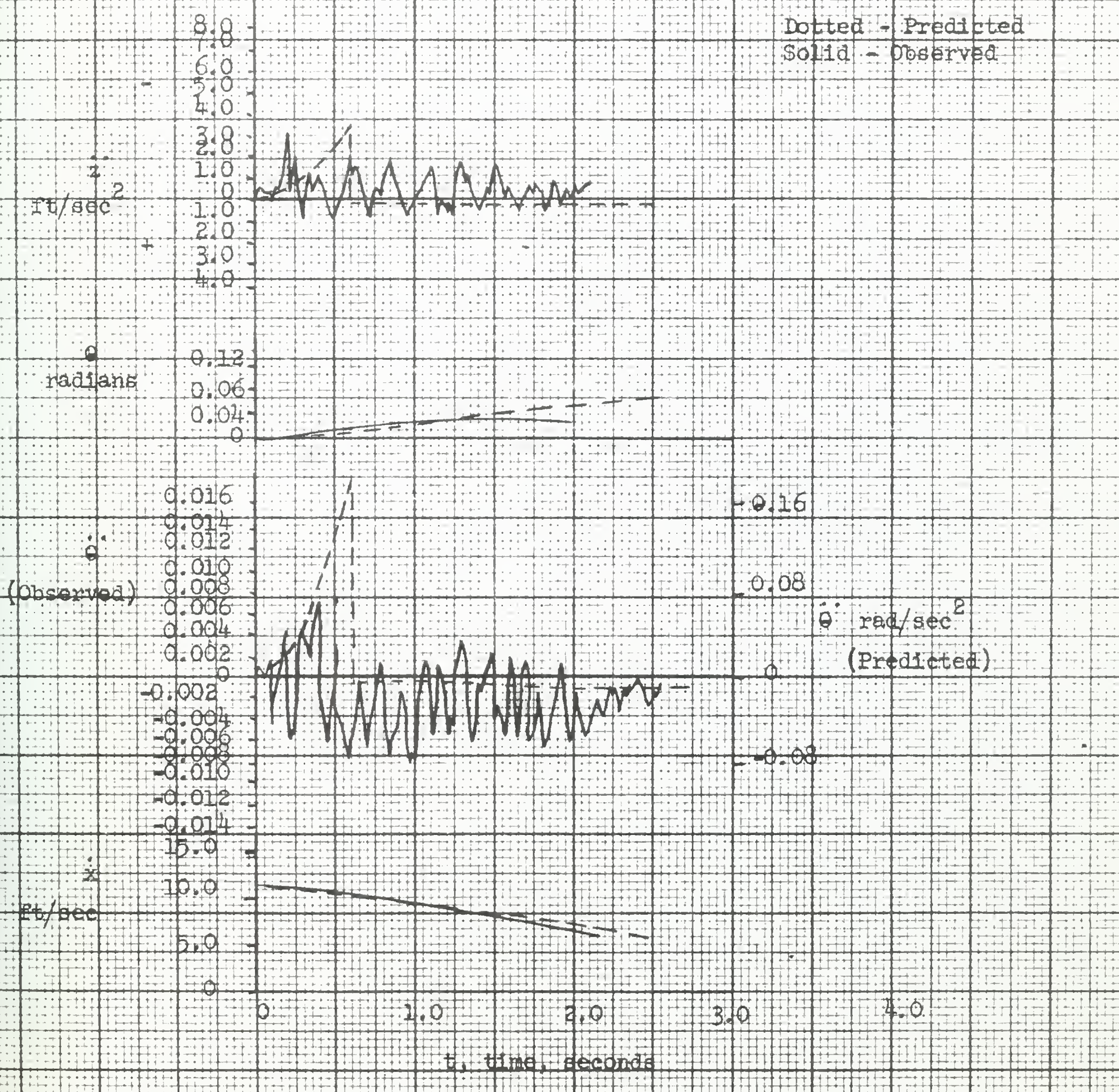


Figure XII

Observed Strain on Bulkhead
and Predicted Force Under Bow
vs
TIME
Run 36B (Parameters given in Table IV)
R m W April 1965
R m W

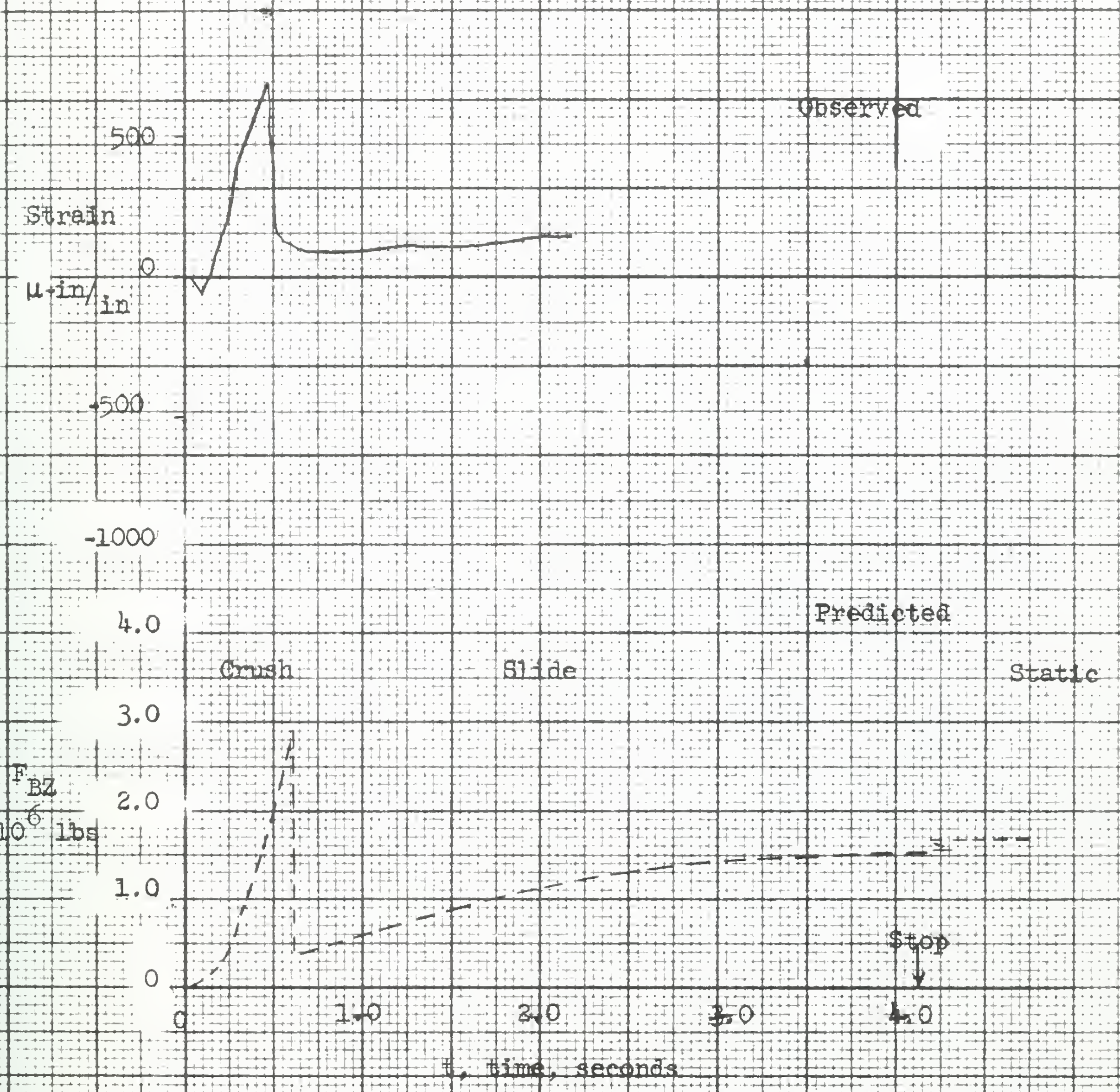


Figure XIII

$\ddot{z}$, θ, $\ddot{\theta}$, and $\dot{x}$ (Predicted and Observed)

vs

TIME

Run 37B (Parameters given in Table IV)
R m W April 1965
RmW

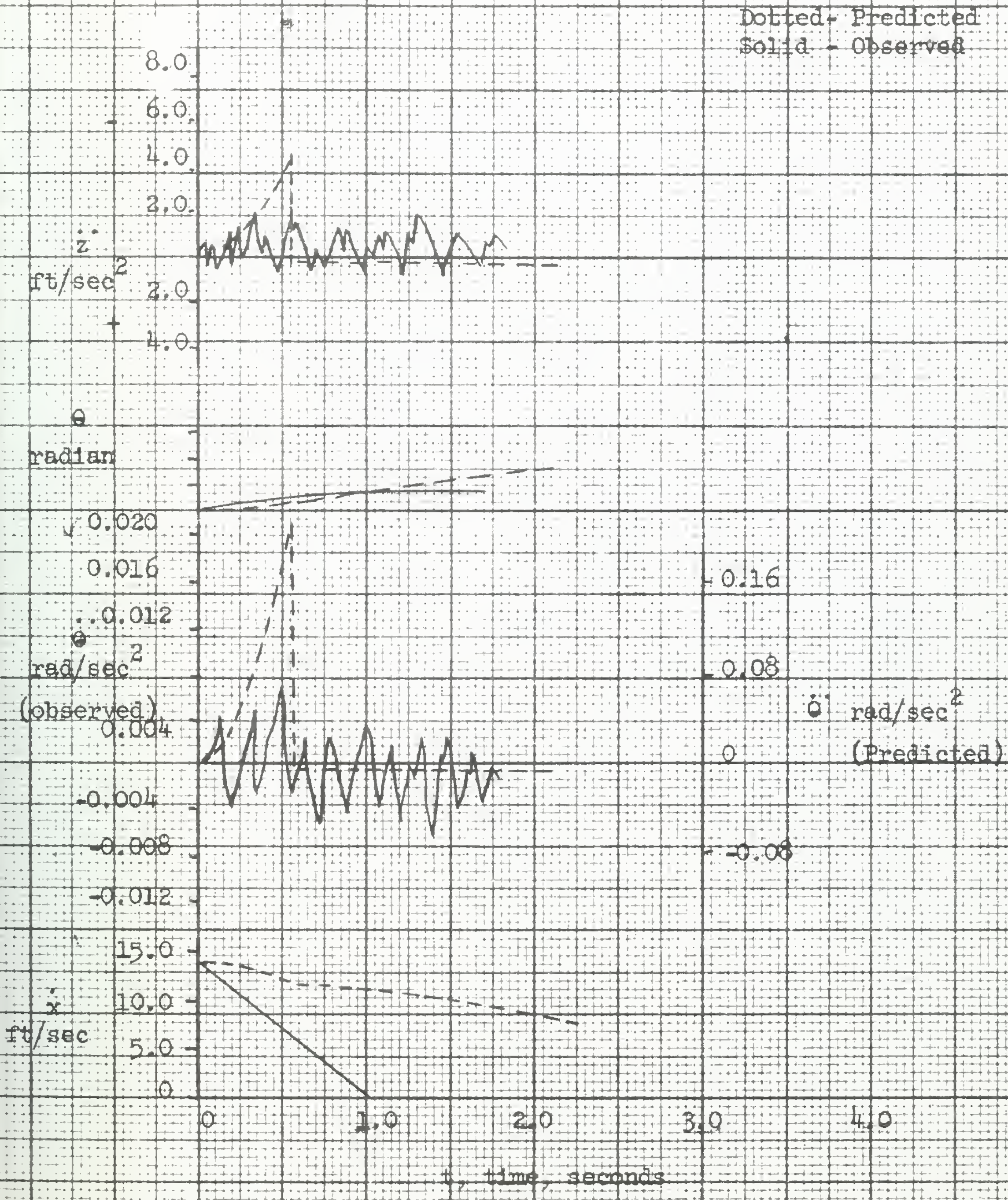


Figure XIV

Observed Strain on Bulkhead
and Predicted Force Under Bow
vs
TIME
Run 37B (Parameters given in Table IV)
r m w April 1965
RmW

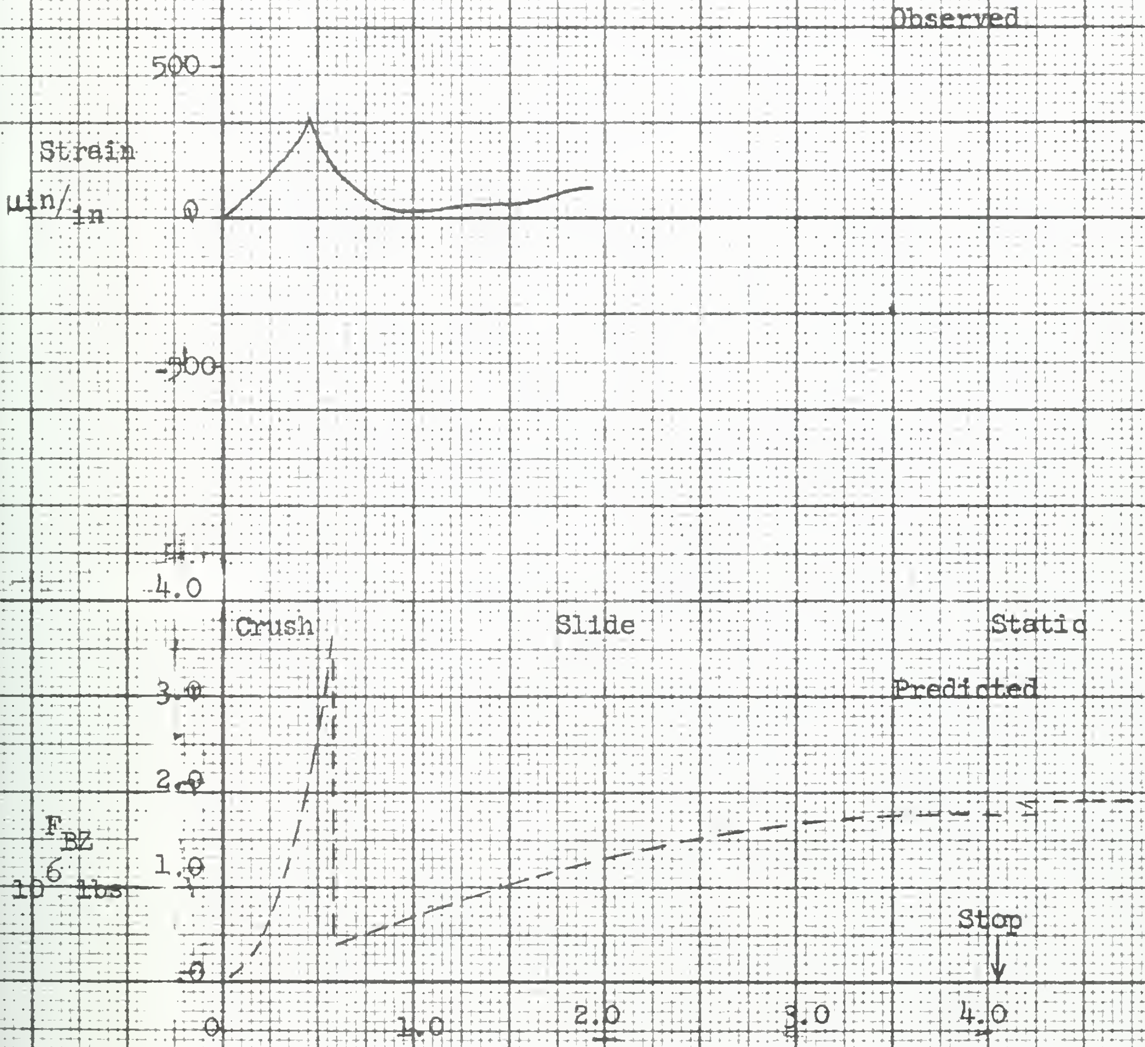


Figure XV

$\ddot{z}$, θ, $\ddot{\theta}$, and $\dot{x}$ (Predicted and Observed)

vs

TIME

Run 38 B (Parameters given in Table IV)
R m W April 1965
RmW

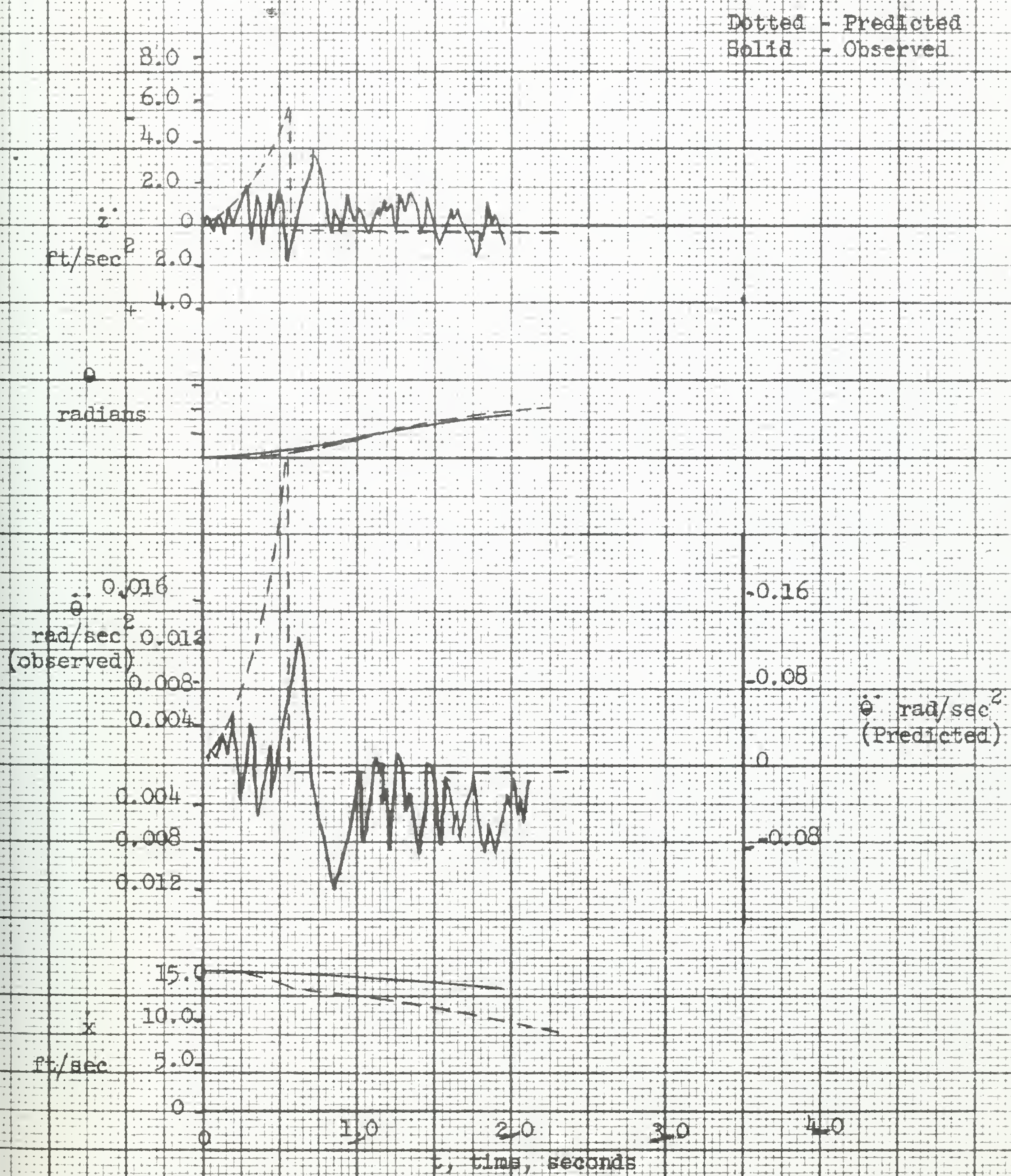
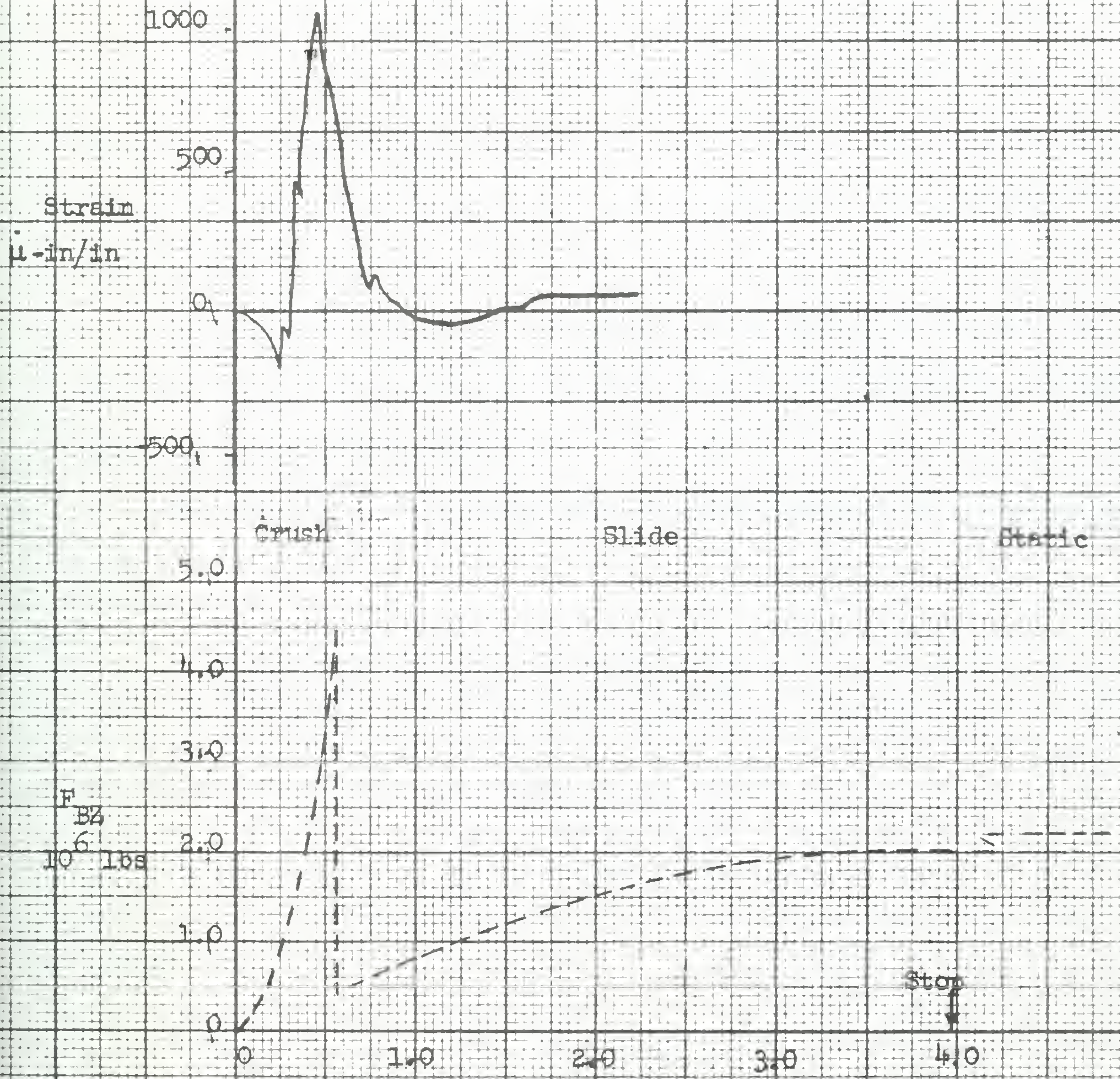


Figure XVI
Observed Strain on Bulkhead
and Predicted Force Under Bow
vs
TIME
Run 38 B (Parameters given in Table IV)
R m W April 1965
RmW



Effect of Variation of Parameters on Bow Force

Figures XVII through XXIX are plots of the icebreaking force (sustained, State 4) as a function of each parameter. A "Wind Class" icebreaker, as indicated in Table V, is used as the parent in each case. The parameters, in many uses, are not independent. The procedure is explained in Chapter II.

In each case the impact velocity is also varied (5,10, 15, 20, 25 ft/sec) and the plots reflect the effect of three increments of impact velocity (5, 15, 25 ft/sec).

The entering argument for each plot is expressed in dimensionless form.

TABLE V

PARAMETERS USED FOR ICEBREAKING CALCULATIONS

(UNLESS OTHERWISE NOTED ON FIGURES)

WIND CLASS

BP = 250.0 ft
H = 25.75 ft
BA = 0.523 rad.
AL = 0.724
CG = -2.40 ft
D = 16.0 ft
GM = 195.6 ft
FS = 0.8

B = 64.0 ft
DIS = 12,100,000 lbs.
SA = 0.886 rad.
CF = -1.25 ft
GK = 23.40 ft
TB = 270,000 lbs
FK = 0.2
SIG = 144,000 lbs/ft^2

GLACIER

BP = 290.0 ft
H = 28.0 ft
BA = 0.523 rad
AL = 0.800
CG = -2.78 ft
D = 16.8 ft
GM = 275.0 ft
FS = 0.8

B = 72.5 ft
DIS = 19,350,000 lbs
SA = 0.886 rad
CF = -1.45 ft
GK = 24.5 ft
TB = 455,000 lbs
FK = 0.2
SIG = 144,000 lbs/ft^2

LENIN

BP = 420.0 ft
H = 30.25 ft
BA = 0.523 rad
AL = 0.800
CG = -4.04 ft
D = 18.8 ft
GM = 545.0 ft
FS = 0.8

B = 90.0 ft
DIS = 35,800,000 lbs
SA = 0.886 rad
CF = -2.10 ft
GK = 27.5 ft
TB = 730,000 lbs
FK = 0.2
SIG = 144,000 lbs/ft^2

Figure XVII

Icebreaking Force of Windclass

vs

Compressive Failure
Stress of Ice

(Parameters given in Table V)

R m W April 1965
Rmw

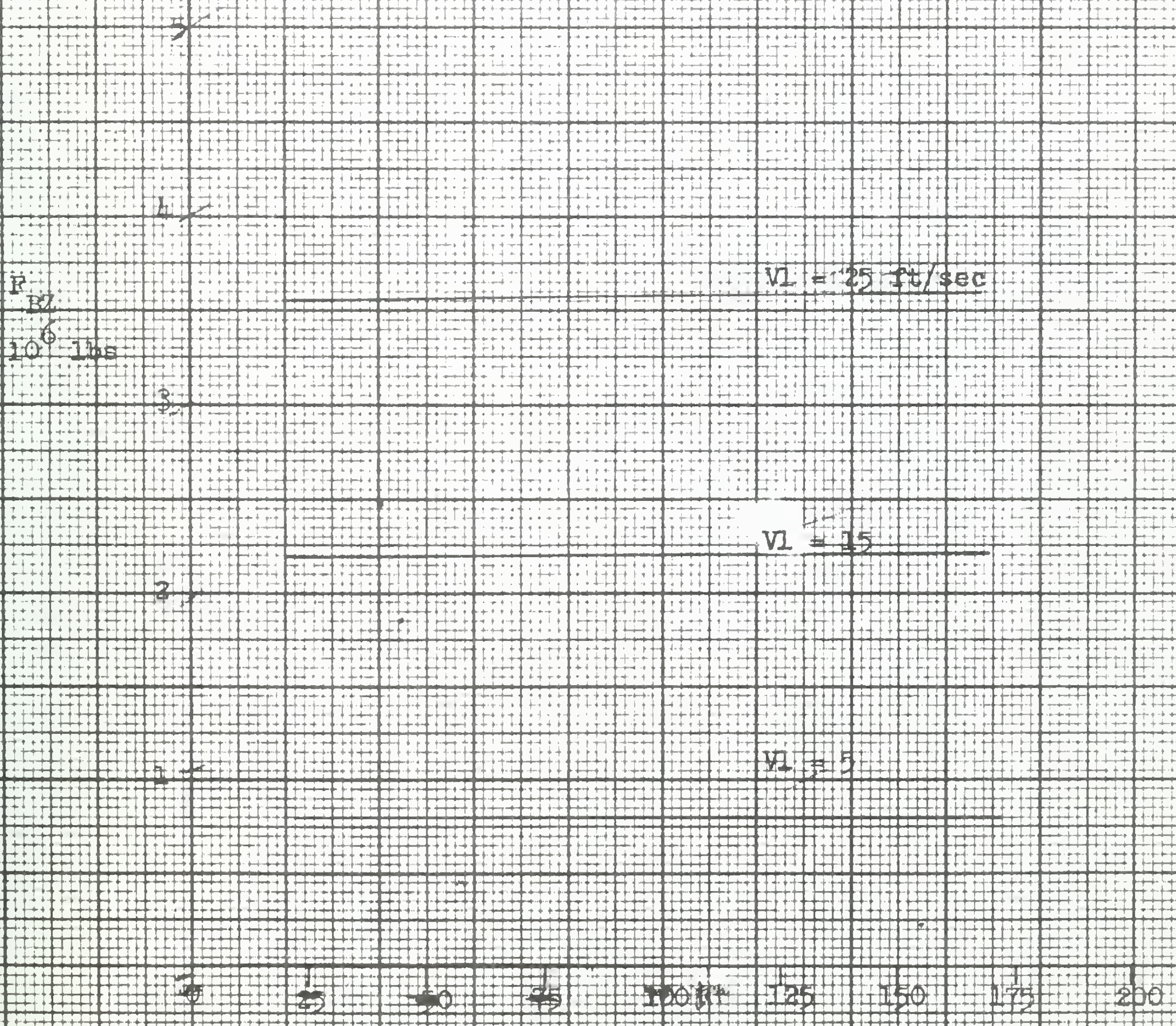


Figure XVIII

Icebreaking Force of Wind Class
vs
Ratio of Height of Thrust Line to Draft
)Parameters given in Table V)

R m W April 1965

R.m.W.

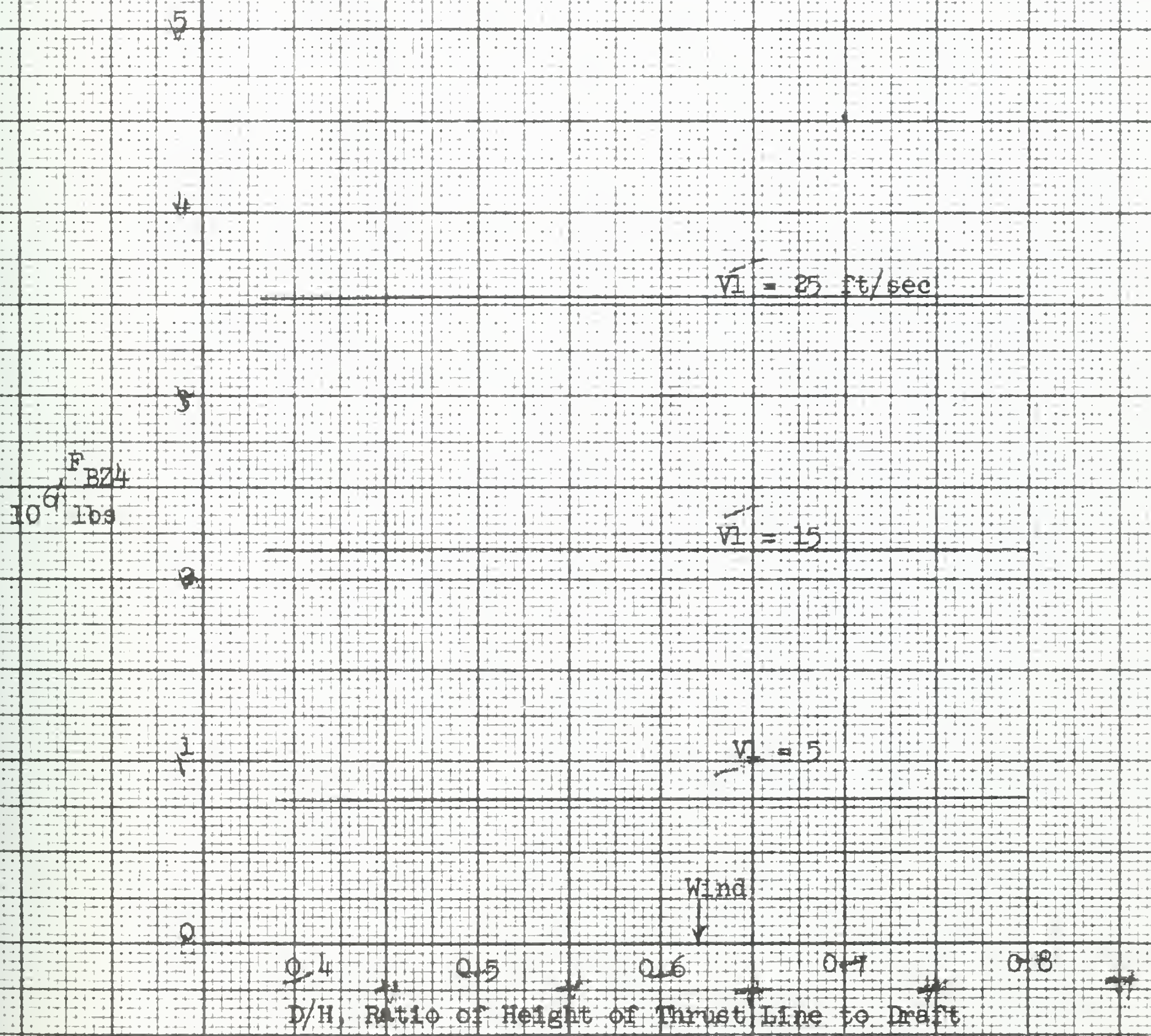


Figure XIX

Icebreaking Force of Wind Class

vs

Ratio of Bollard Thrust (lbs) to Displacement (lbs)

(Parameters given in Table V)

R m W April 1965

RmW

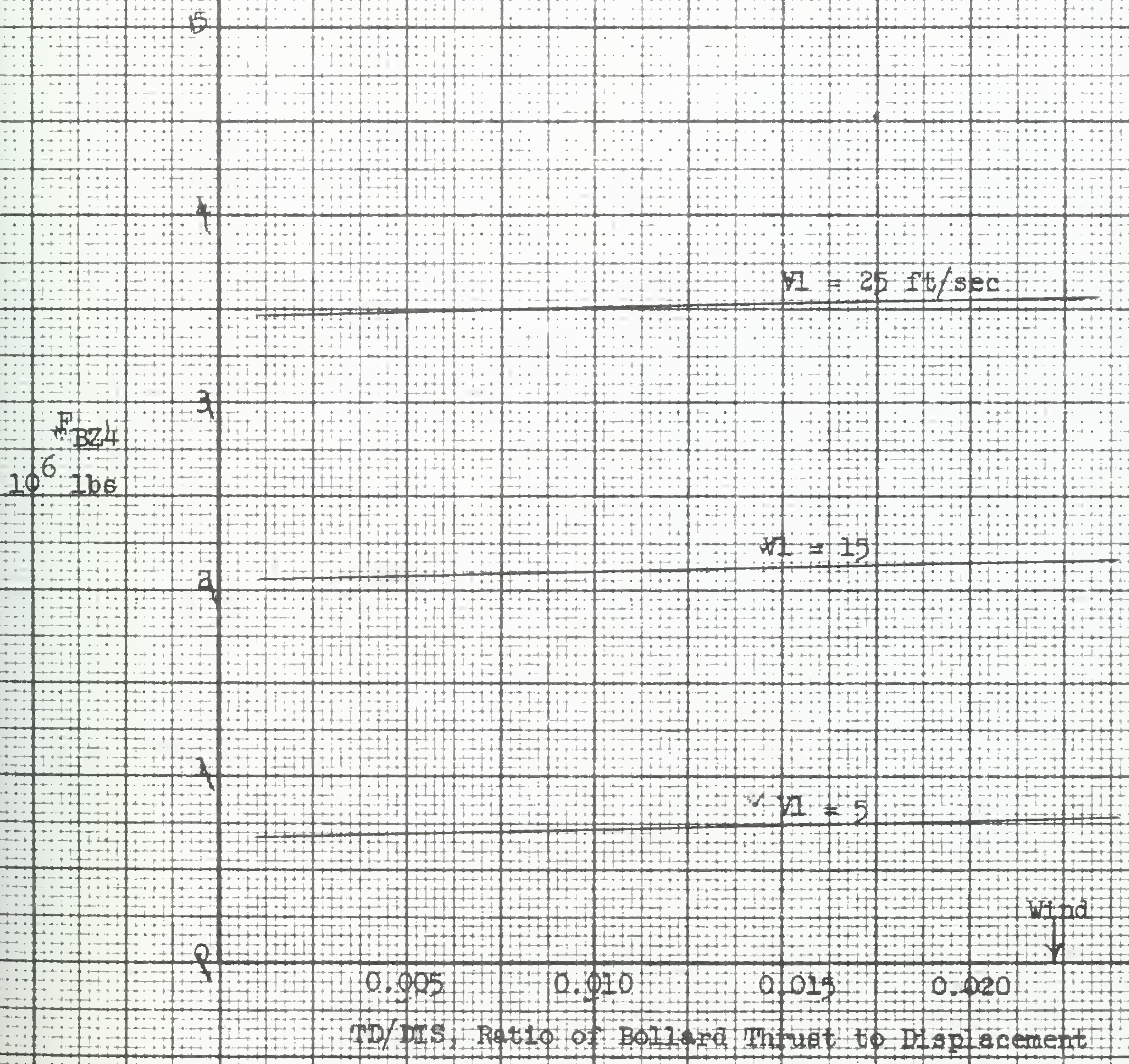


Figure XX

Icebreaking Force of Wind Class

vs

Ratio of Long. Position of Center of Gravity to Length (Parameters given in Table V)

RMW 4/65

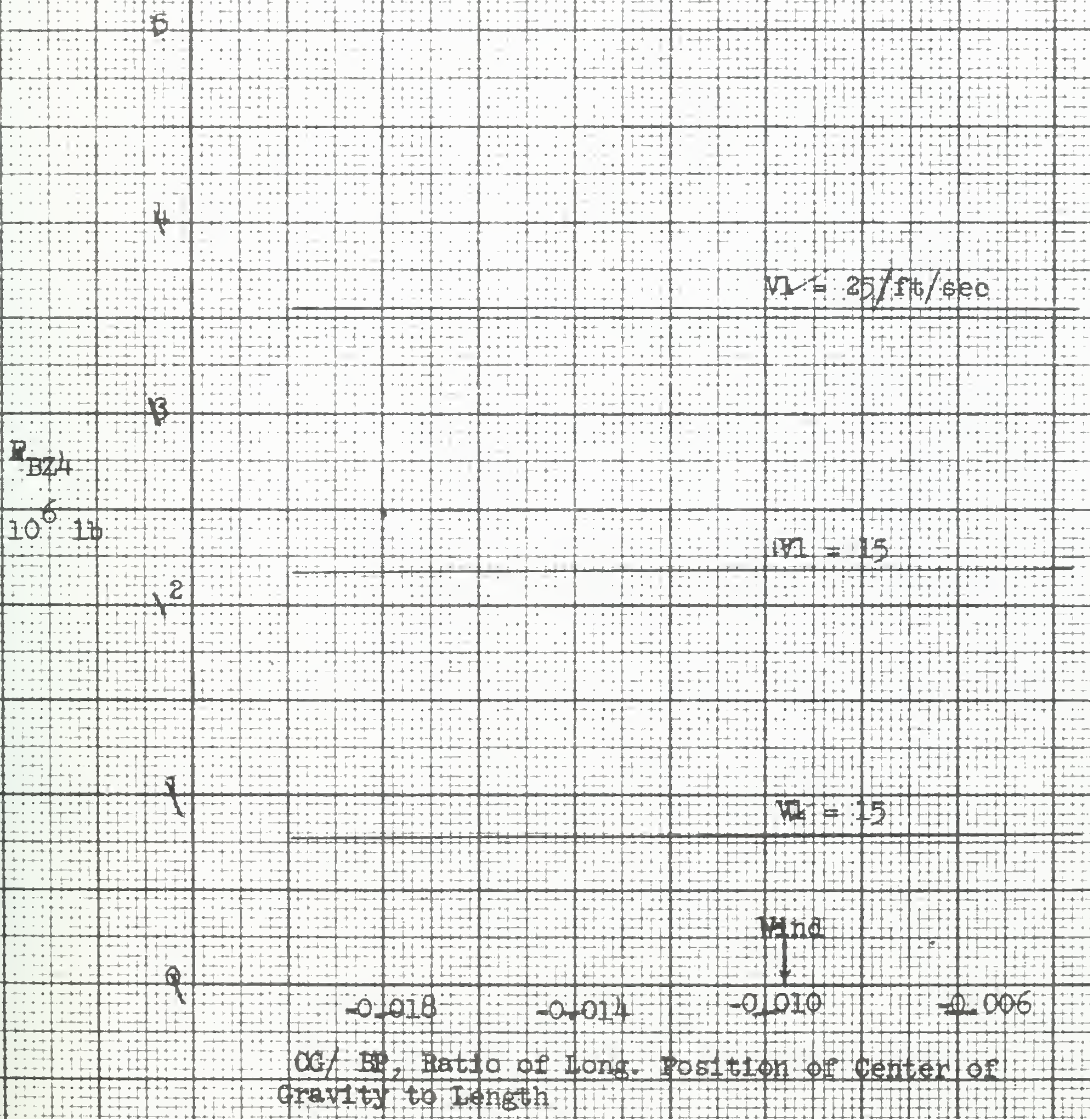


Figure XXI

Icebreaking Force of Wind Class

vs

Ratio of Long. Position of Center
of Flotation to Length
(Parameters given in Table V)

R m W April 1965

RmW

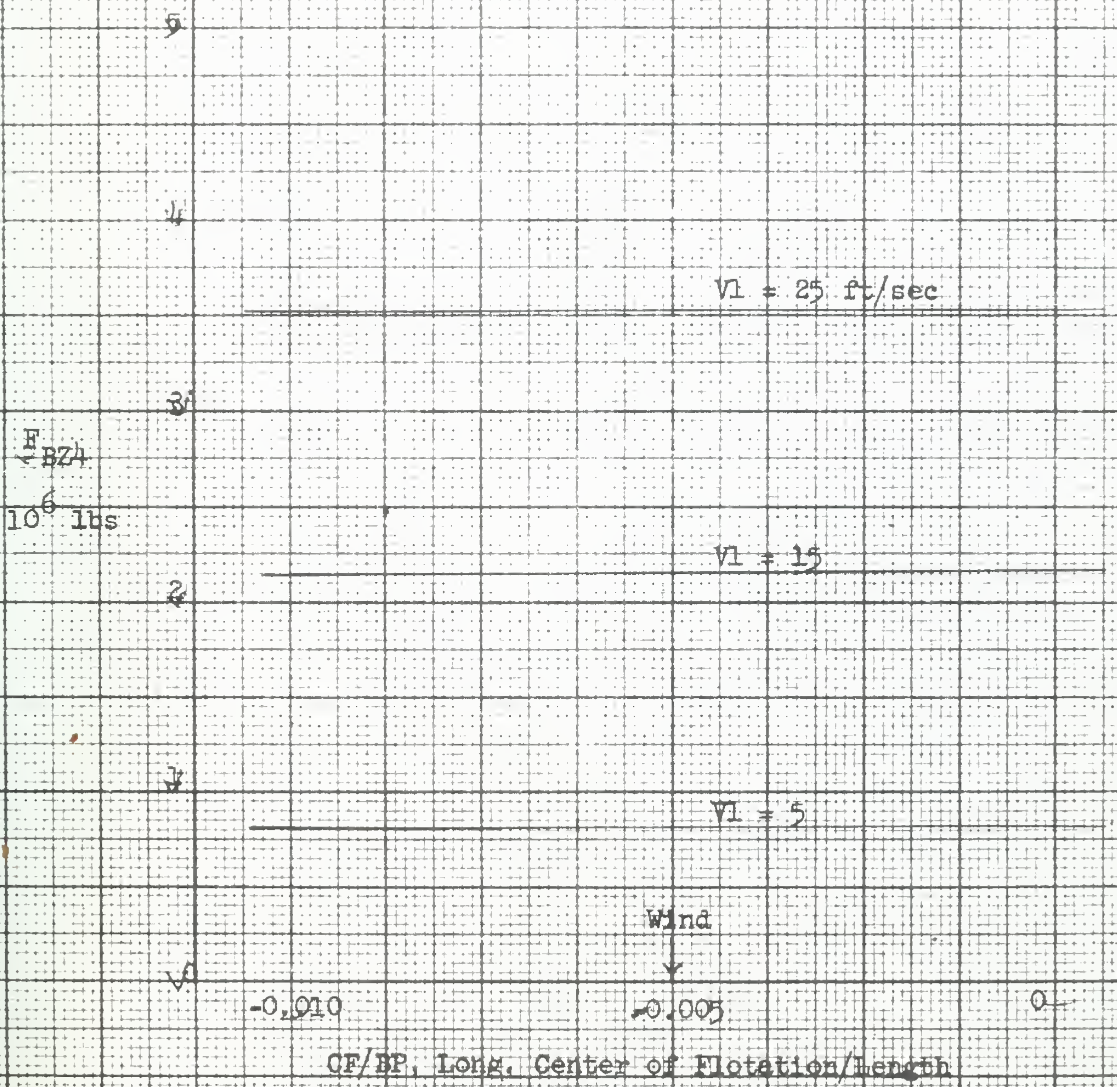


Figure XXII

Icebreaking Force of Wind Class

vs

Length-to-Beam Ratio
(Parameters given in Table V)
R m W April 1965

Rm W

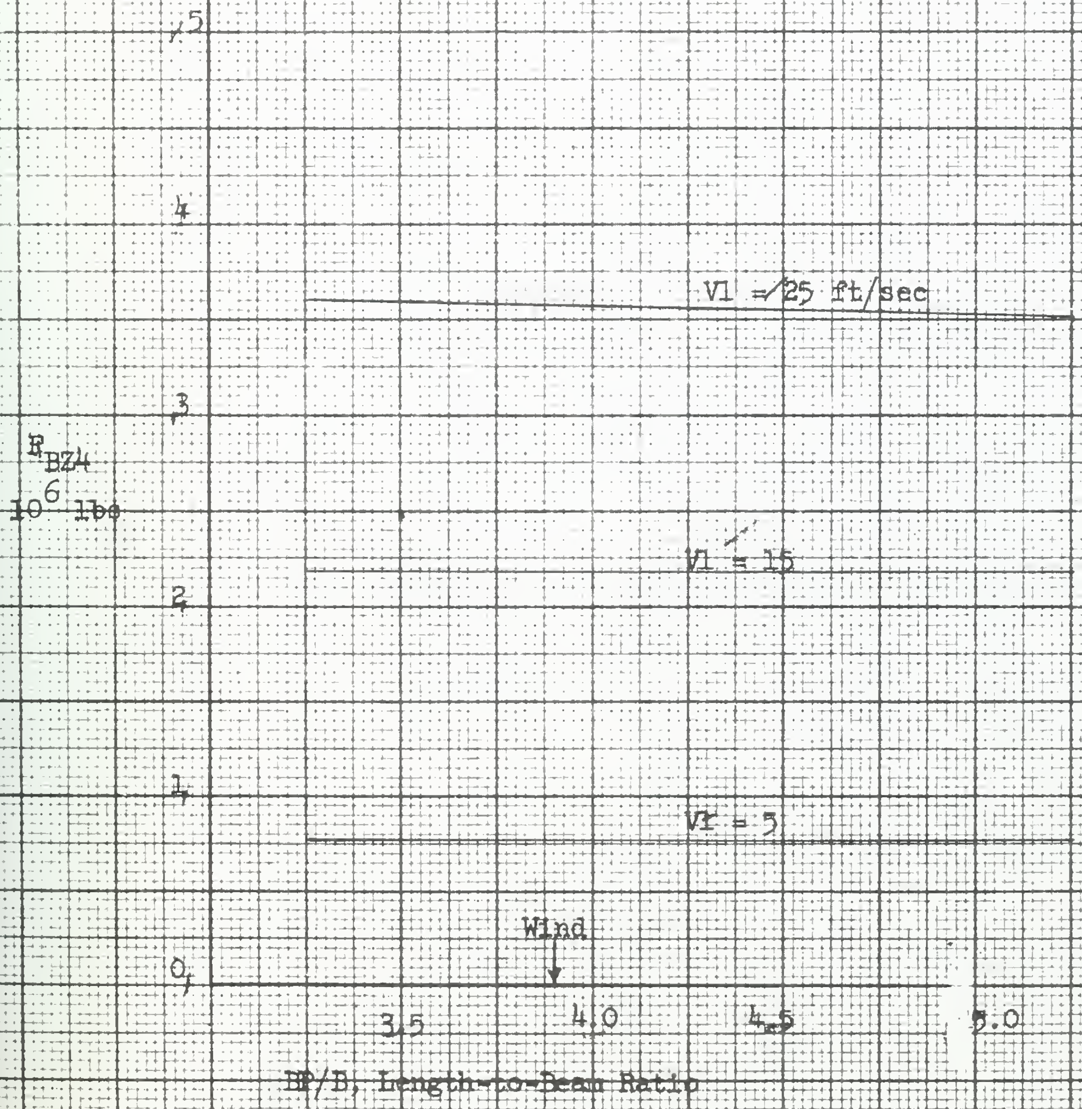


Figure XXIII
Icebreaking Force of Wind Class
vs
Ratio of Height of Center of
Gravity to Draft
(Parameters given in Table V)
R m W April 1965
R m W

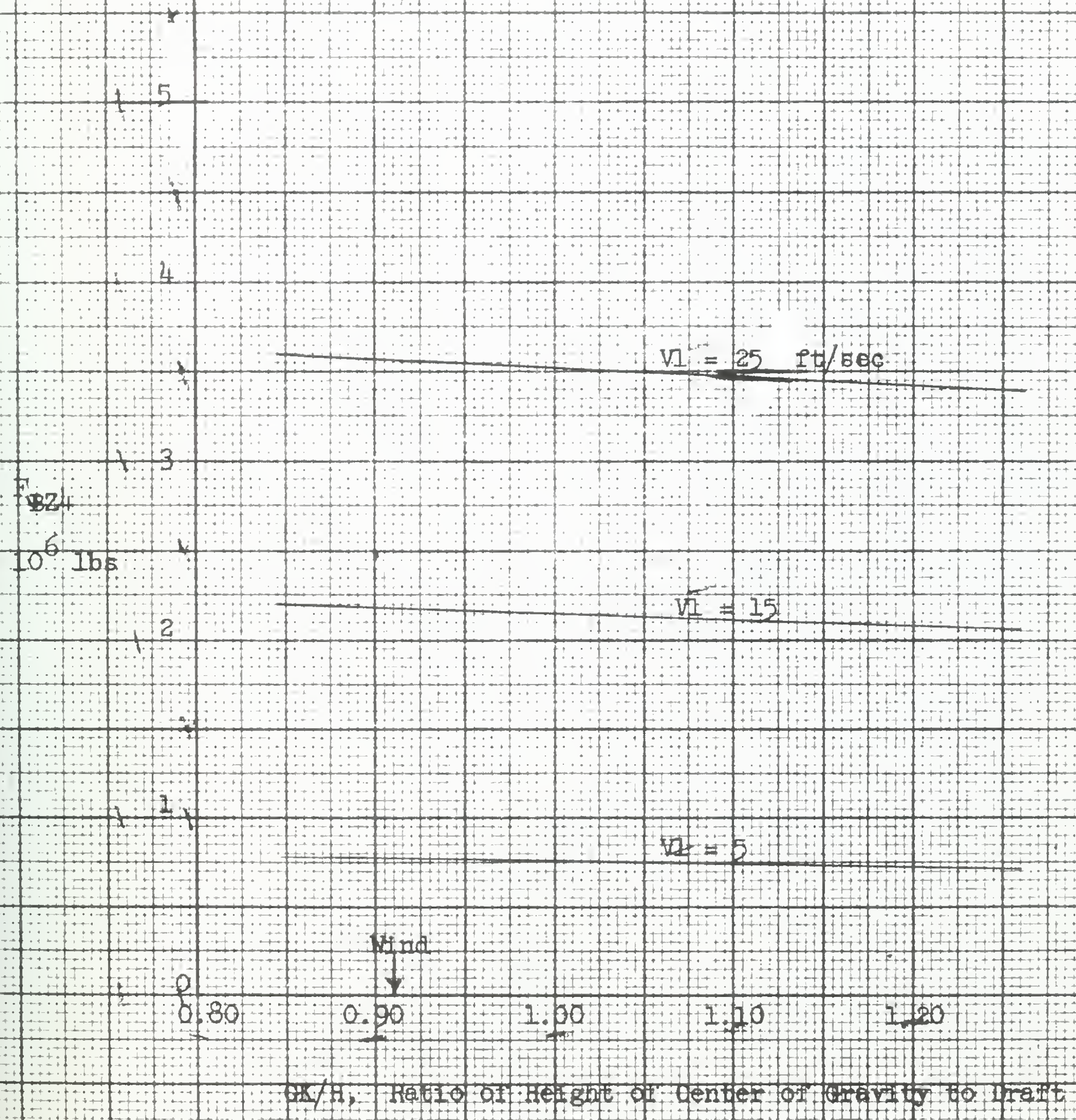


Figure XXIV

Icebreaking Force of Wind Class

vs

Beam-To-Draft Ratio

(Parameters given in
Table V)

R m W April 1965

RmW

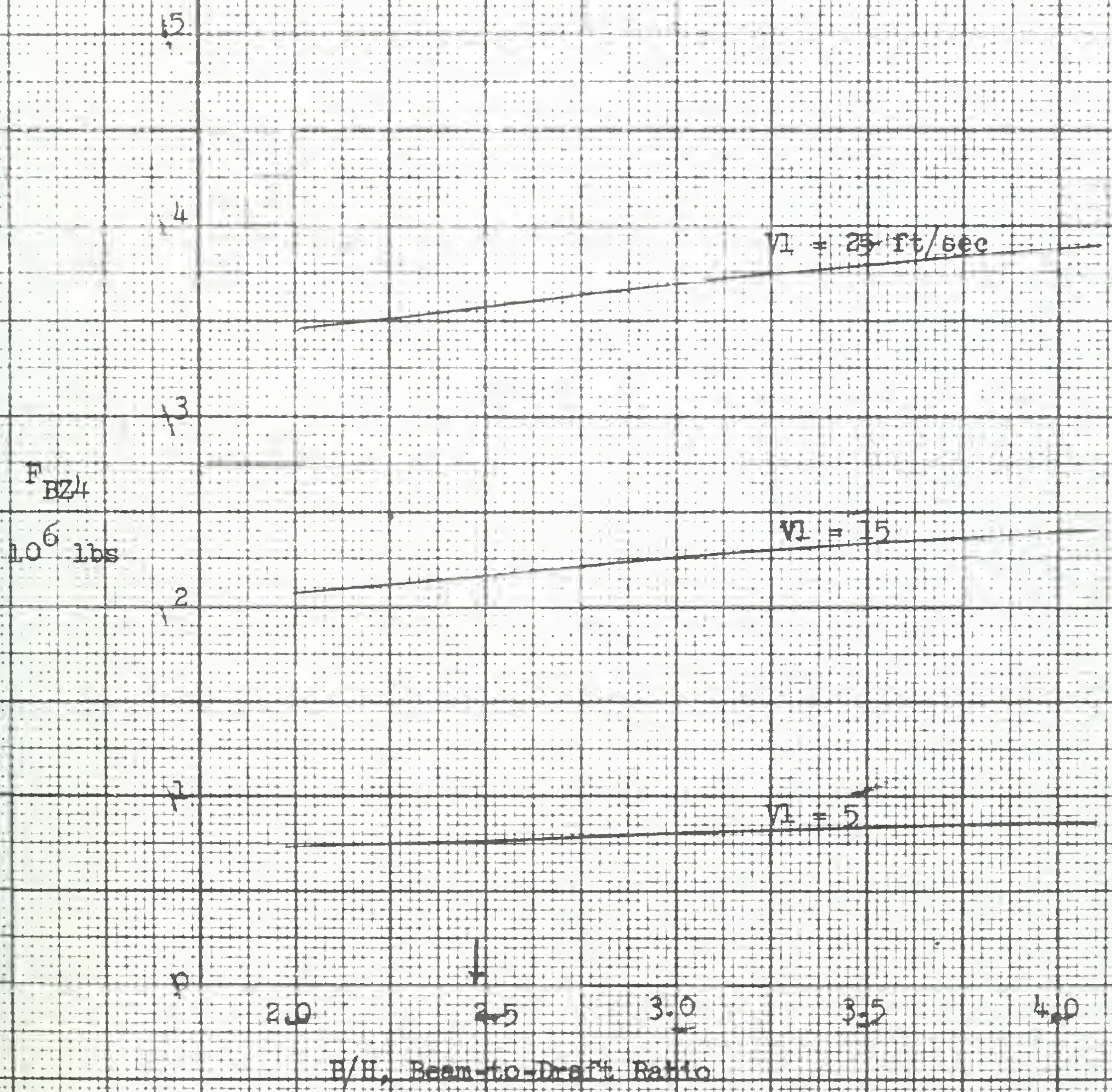


Figure XXV

Icebreaking Force of Wind Class

vs

Waterplane Coefficient
(and Long. GM-to-Length Ratio)

(Parameters given in Table V)

RMW April 1965
Rmw

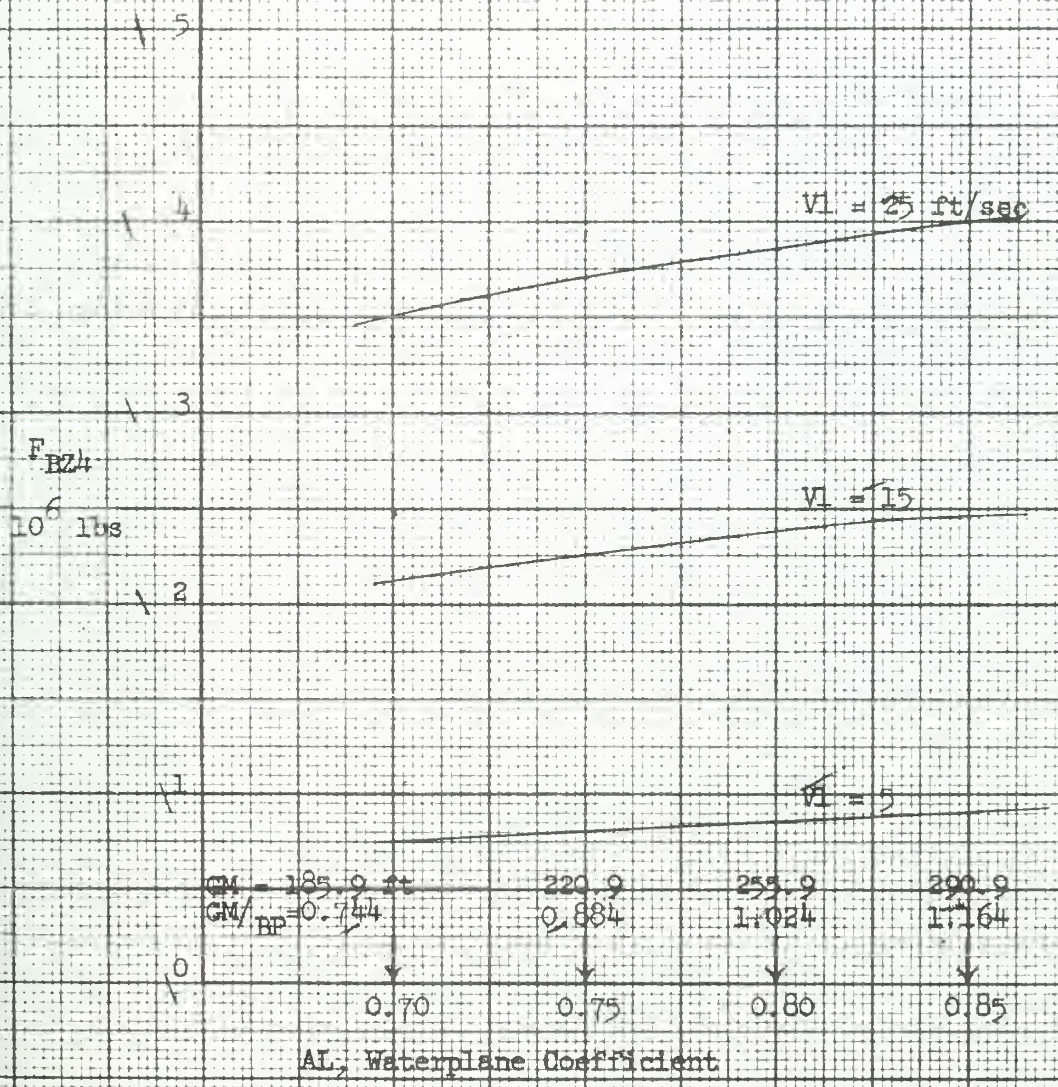


Figure XXVI

Icebreaking Force of Wind Class

vs

Block Coefficient
(and Displacement)

(Parameters given in Table V)

RMW April 1965

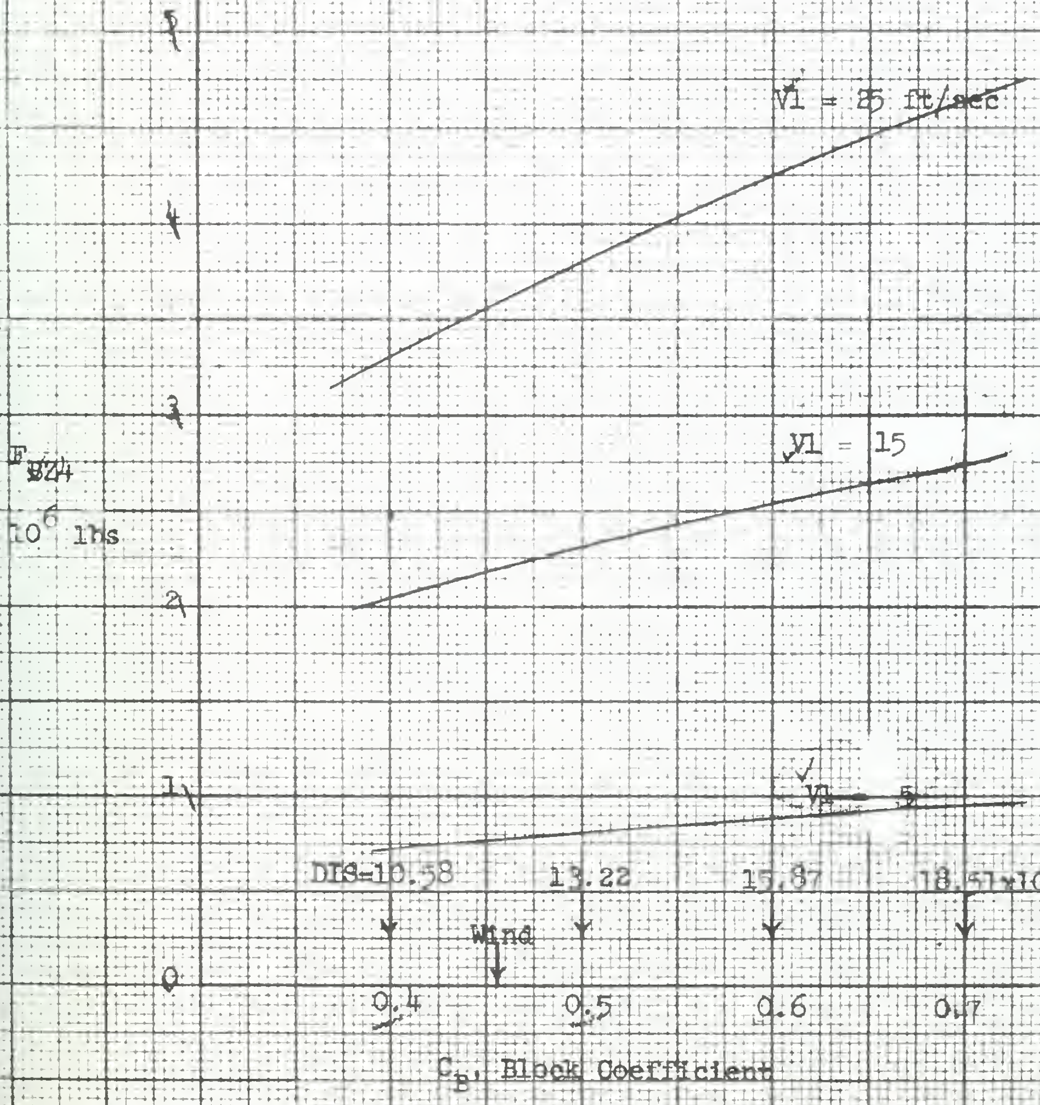


Figure XXVII

Icebreaking Force of Wind Class
vs
Spread Angle Complement
(Parameters given in Table V)
RMW April 1965

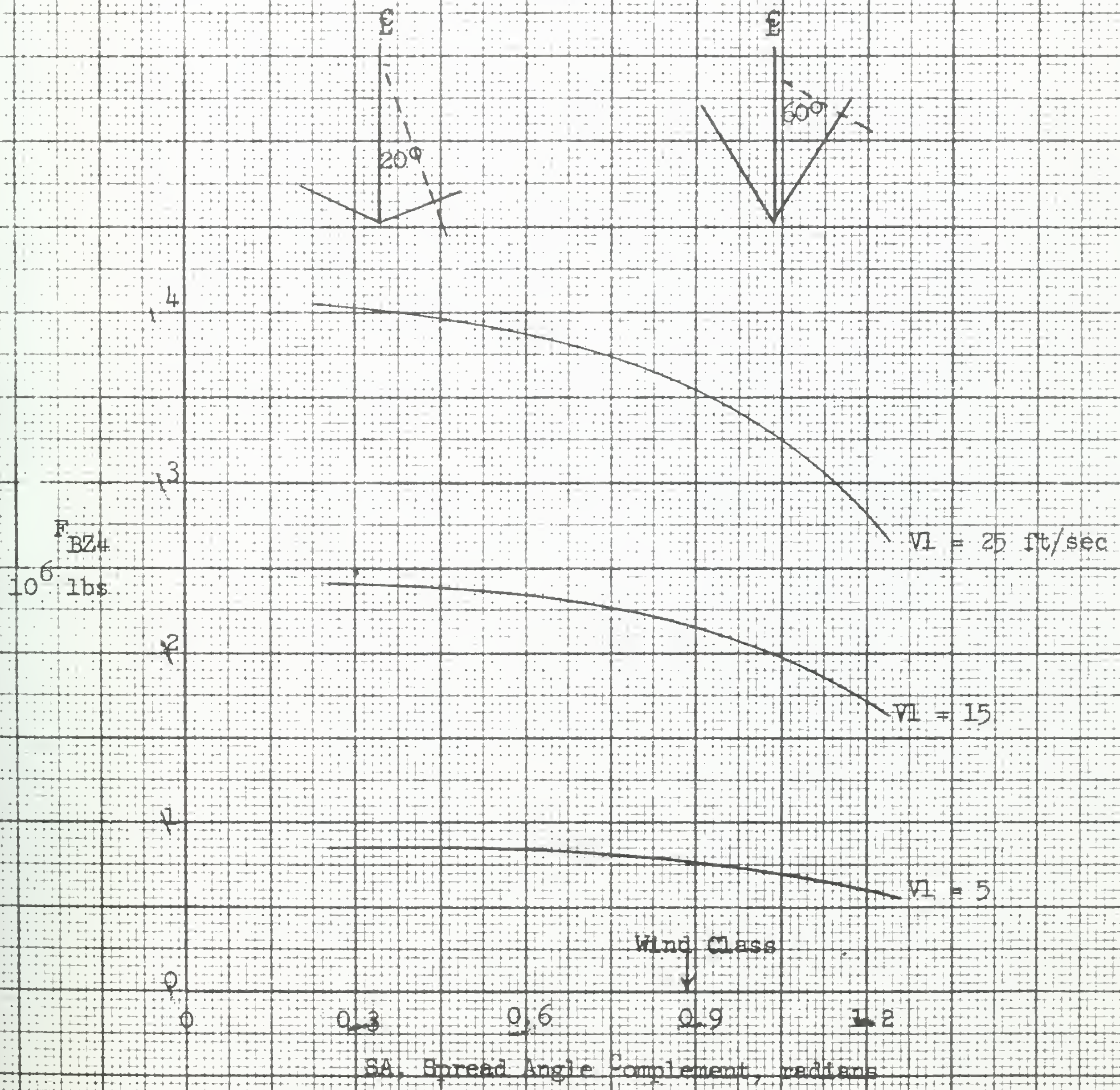


Figure XXVIII

Icebreaking Force of Wind Class

vs

Coefficient of Kinetic Friction

(Parameters given in Table V)

RMW April 1965

RmW

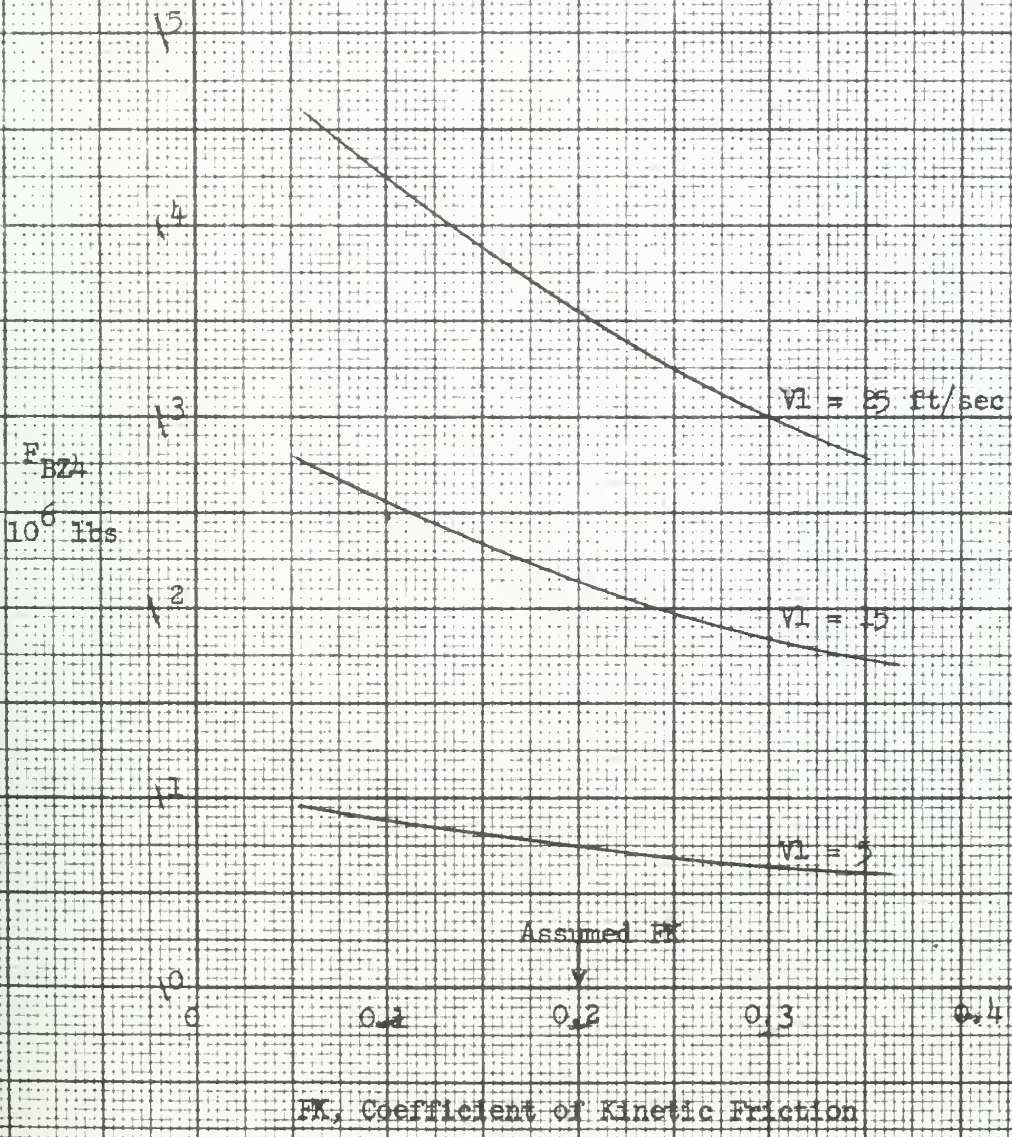


Figure XXIX

Icebreaking Force of Wind Class

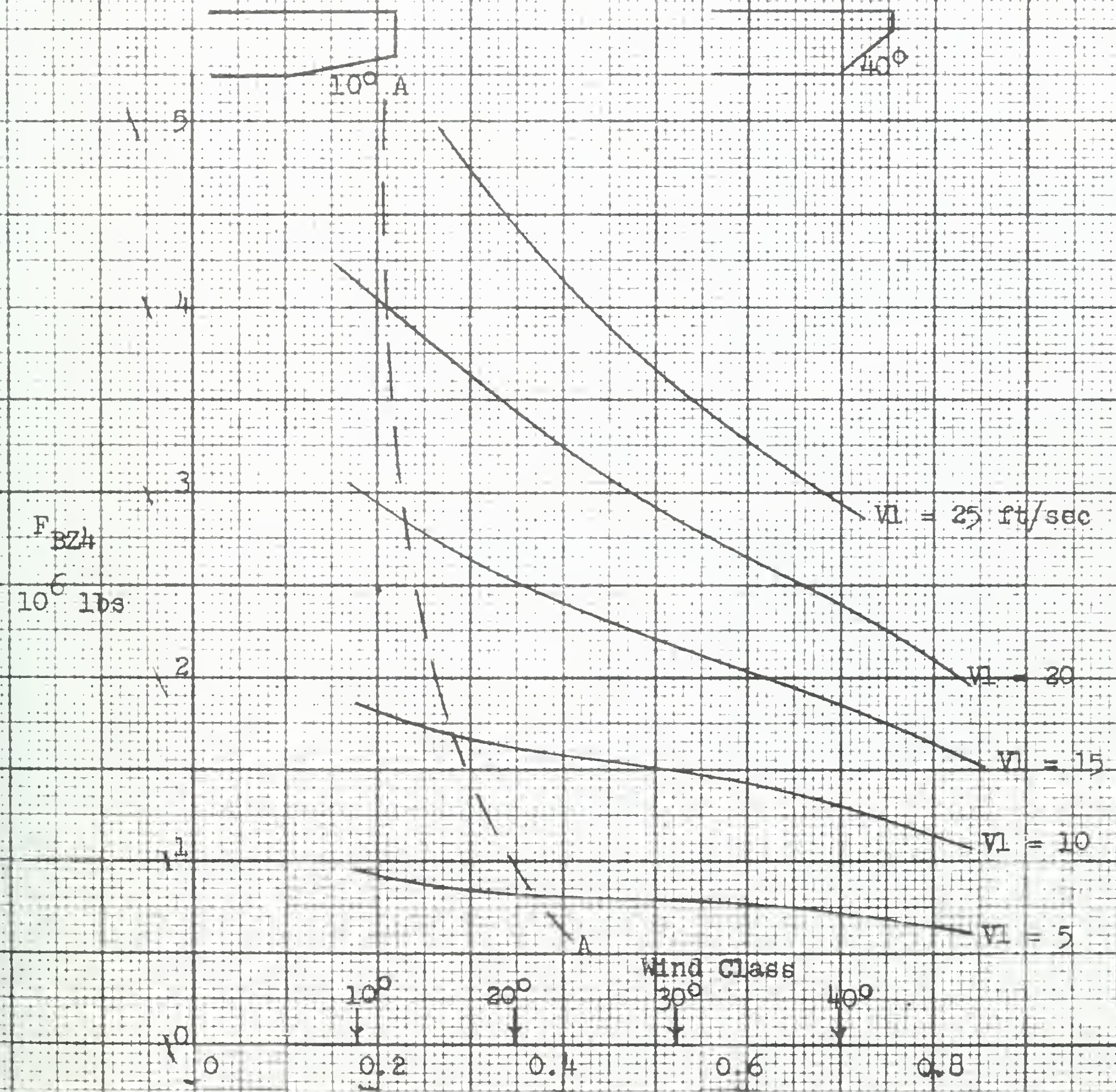
vs

Bow Angle

(Parameters given in Table V)

RMW April 1965

RMW



Effect of Displacement and Impact Velocity on Bow Force

Figure XXX is a plot of icebreaking force as a function of displacement for three different impact velocities. The displacements represent the "Wind Class", "Glacier Class", and "Lenin Class". In reality only the nine points plotted are the direct result of calculation. The curves have been drawn in to represent the trend.

Figure XXXI is a plot of icebreaking force as a function of impact velocity for the three above-mentioned classes of icebreakers.

Figure XXXII is a plot of icebreaking force as a function of displacement for an impact velocity of 15 ft/sec. There are three curves; each curve represents icebreakers which are geometrically similar to the parent icebreaker indicated.

The parameters used for the parent icebreakers are given in Table V.

Effect of Variation of Parameters on "White Ratio"

As indicated in the procedure, since the icebreaking force is approximately linear with respect to impact velocity and displacement, it appears useful to divide the icebreaking force by displacement times velocity (which is the "White Ratio").

Figures XXXIII through XXXX are plots of this ratio as a function of various parameters. These are based on an impact velocity of 15 ft/sec although other velocities give approximately the same value. The three major classes are each plotted so that similar tendencies and magnitudes

may be illustrated.

The parameters for the three parent icebreakers are given in Table V.

Extracting Thrust

Figures XXXXI and XXXXII are plots of the ratio of extracting thrust to the maximum (forward) bollard thrust available as a function of bow angle, coefficient of static friction, and impact velocity. In a sense, Figure XXXXII is a set of cross-curves of Figure XXXXI.

Figure XXXXIII is a plot of the ratio of extracting thrust to the bollard thrust as a function of the spread angle complement (for various impact velocities of the three major classes).

The parameters used are given in Table V.

Figure XXX

Icebreaking Force
vs
Displacement of

Lenin
Glacier
Wind
(Parameters given in Table V)
RMW April 1965
RmW

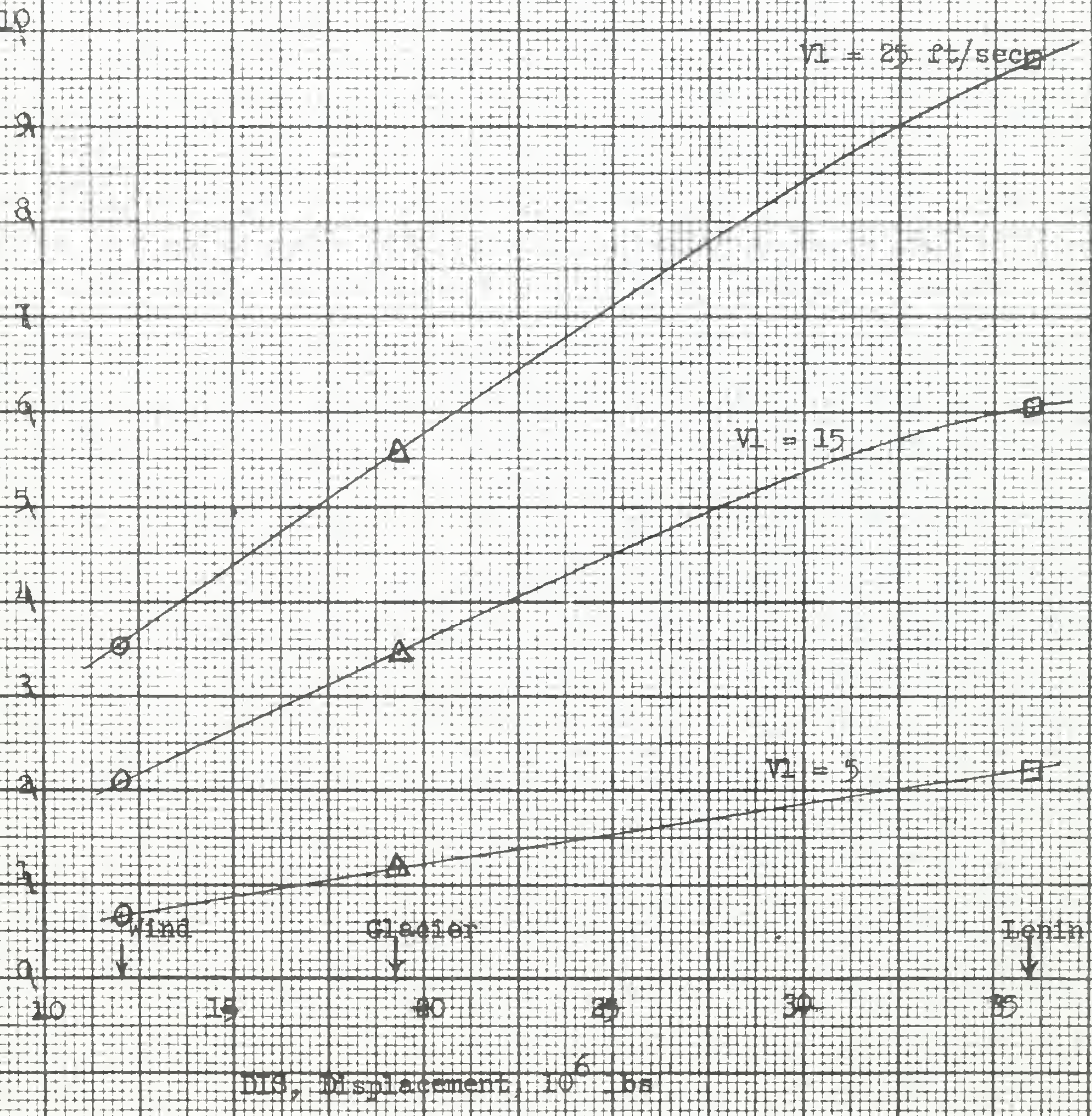
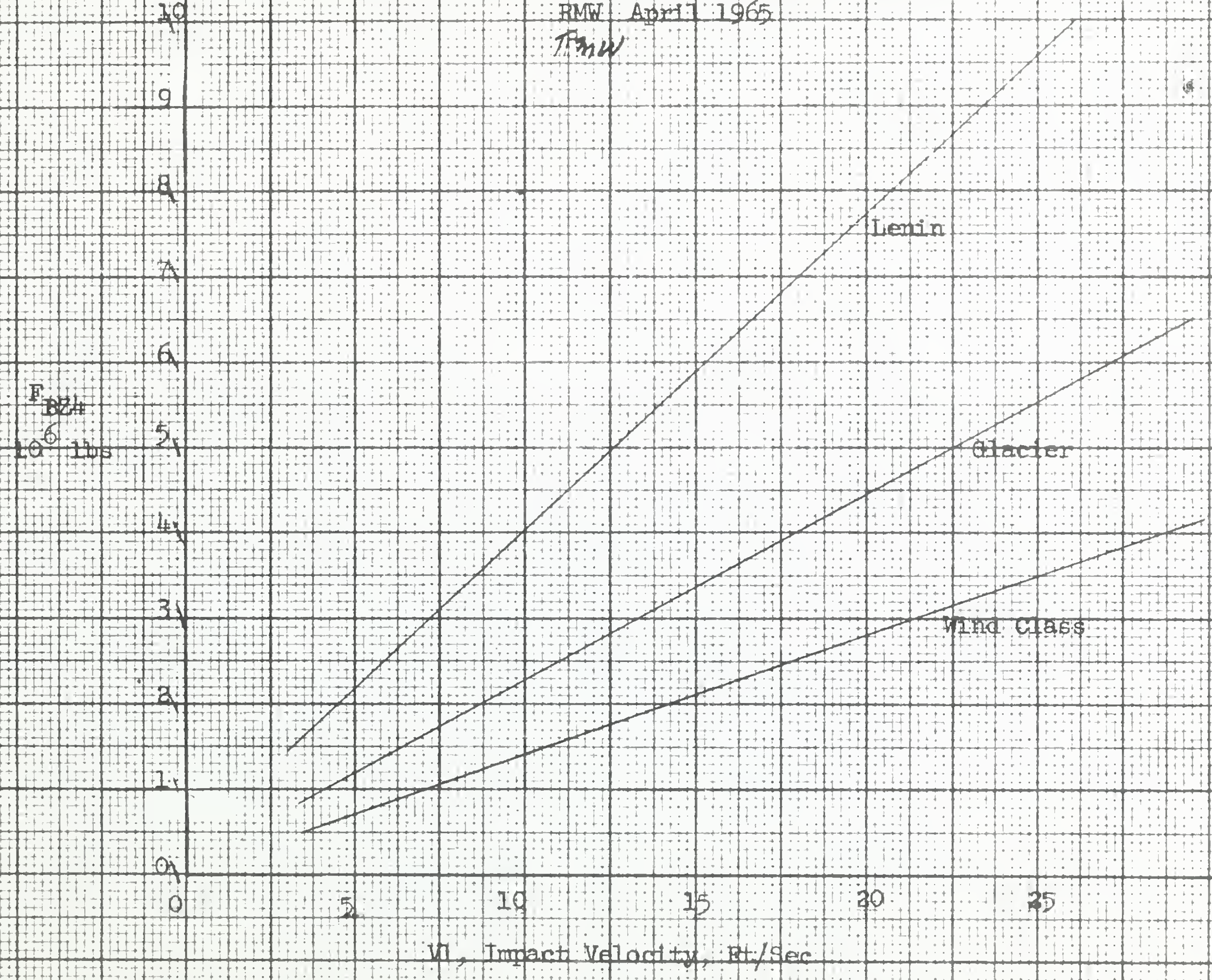


Figure XXXI
Icebreaking Force
vs
Impact Velocity
(Parameters given in Table V)

RMW April 1965



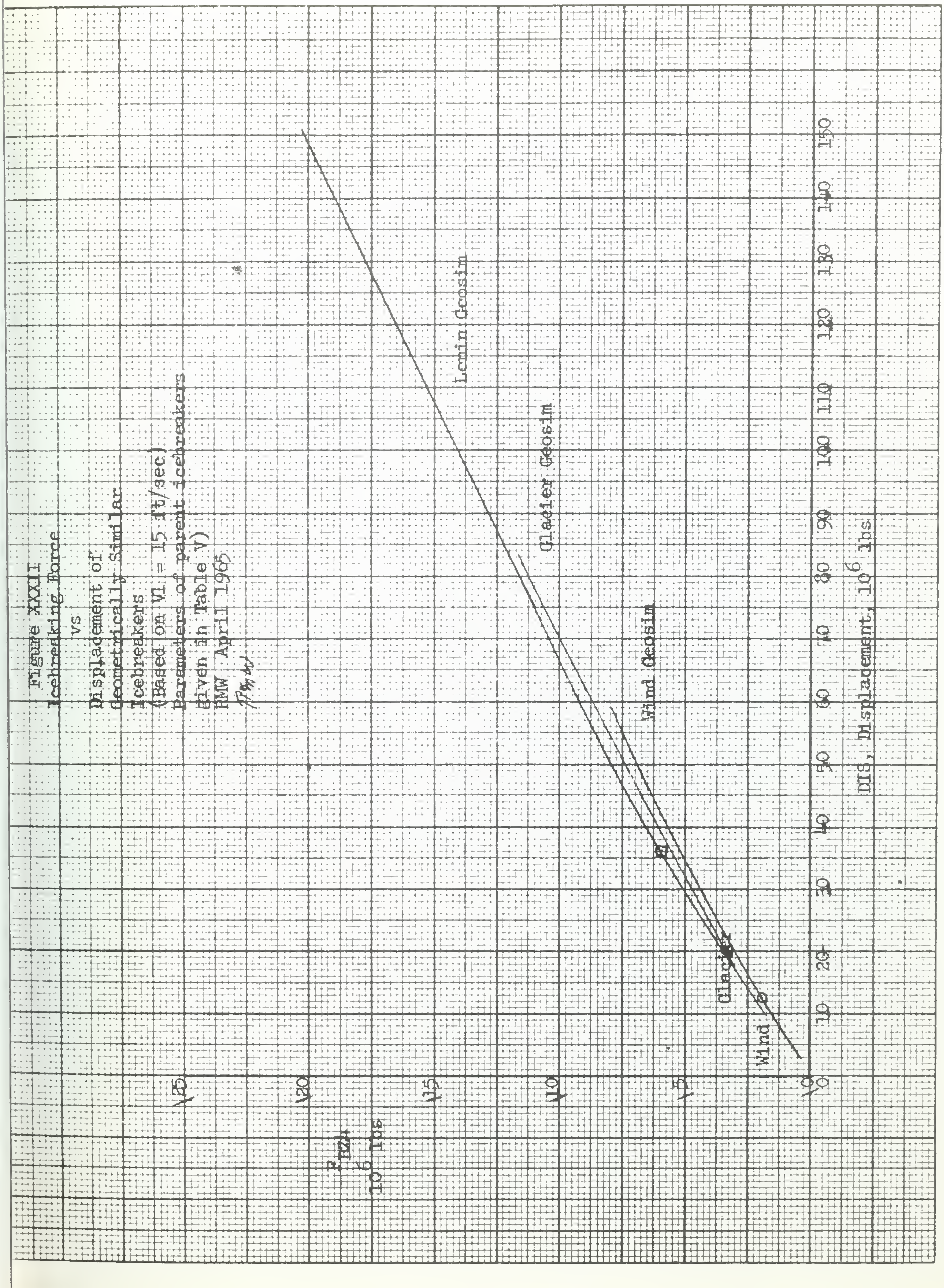


Figure XXXII
Icebreaking Force
vs
Displacement of
Geometrically Similar
Icebreakers
(Based on VI = 15 ft/sec)
Parameters of parent icebreakers
given in Table V)
RMW April 1965

Figure XXXIII
White Ratio
vs
Beam-To-Draft Ratio
(Based on V1 = 15 ft/sec)
(Parameters given in Table V)
RMW April 1965
R.M.W

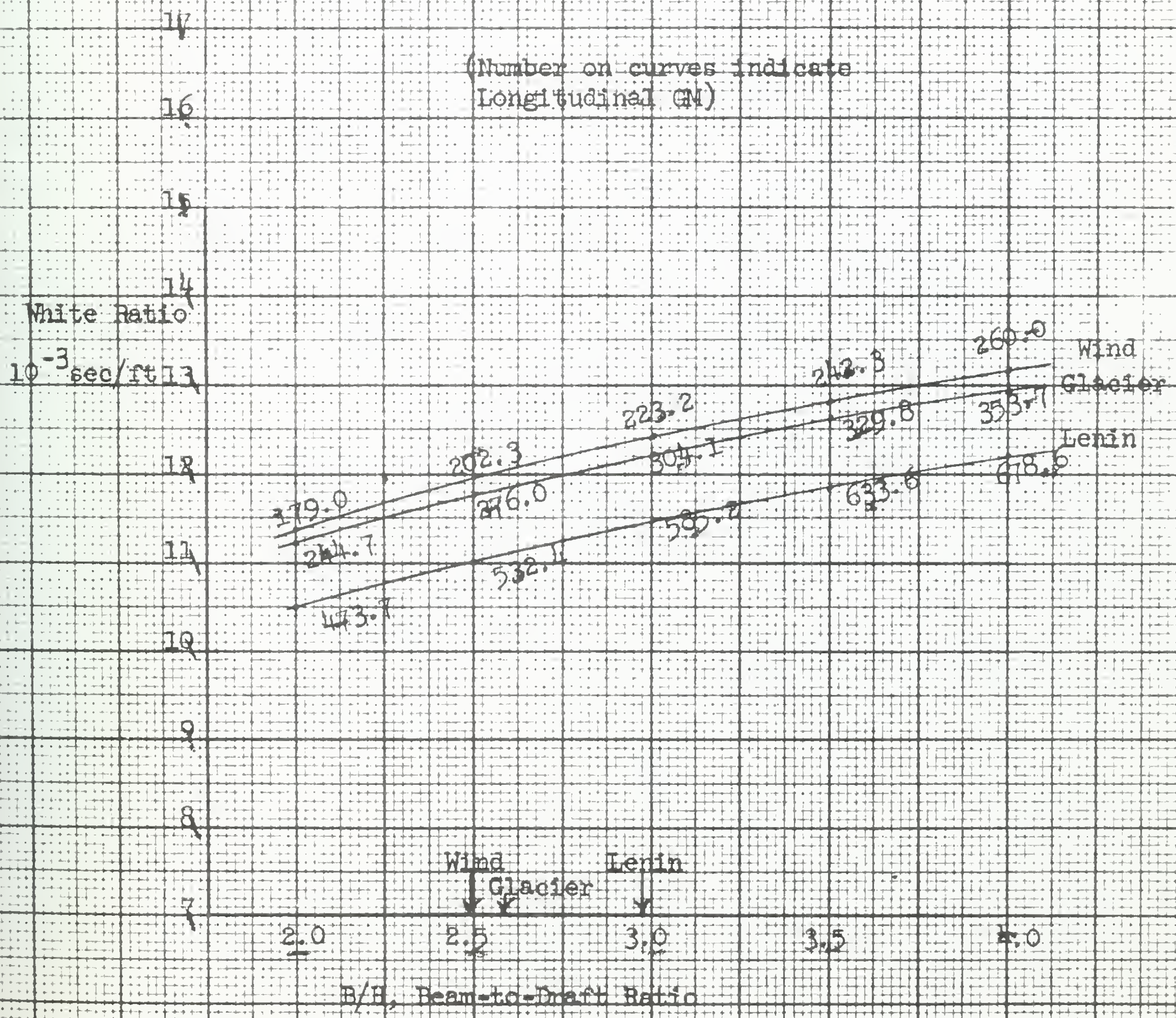


Figure XXXIV
White Ratio
vs
Waterplane Coefficient
(and Long. GM-to-Length Ratio)
(Based on Vl = 15 ft/sec)
(Parameters given in Table V)
RMW April 1965

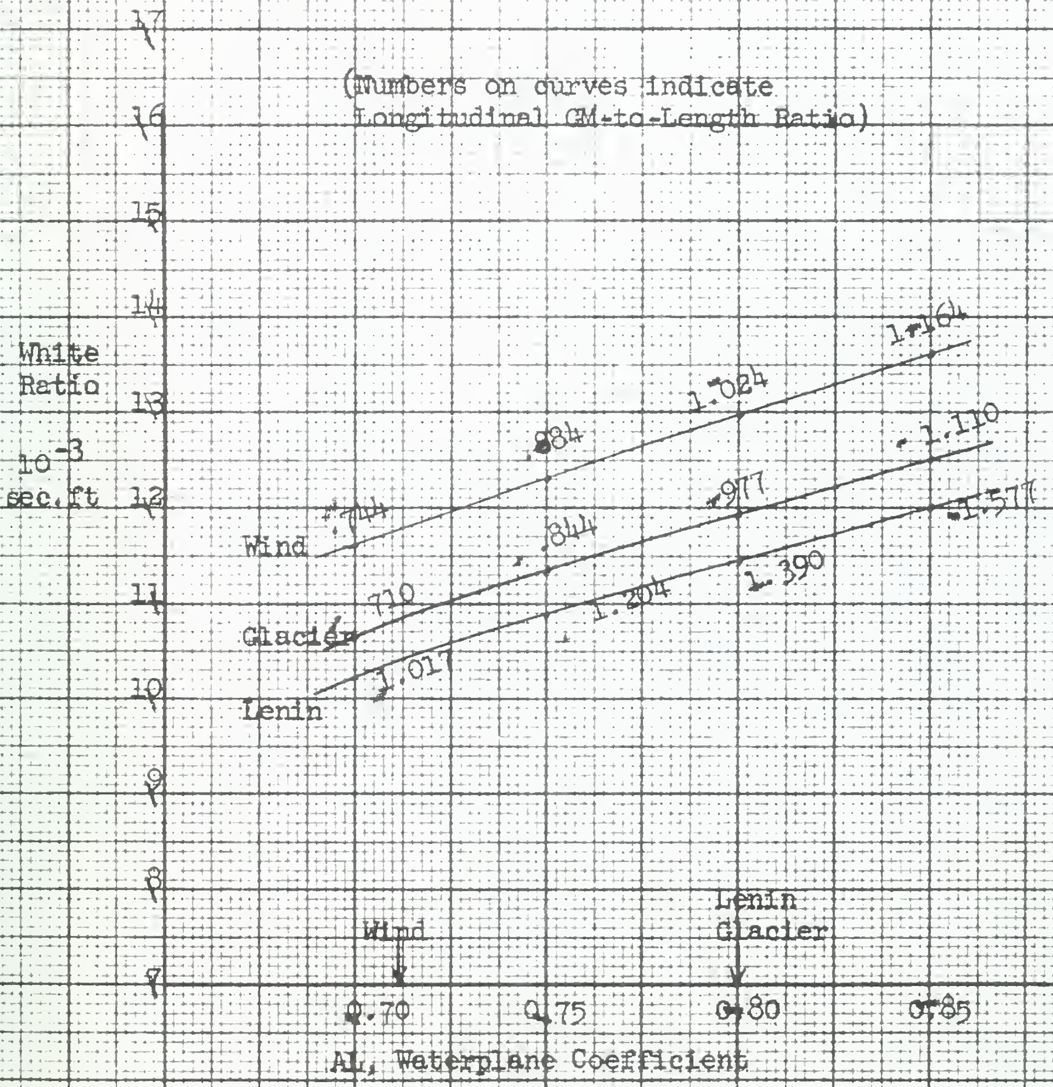


Figure XXXV

White Ratio
vs
Block Coefficient
(And Displacement)
(Based on Vl = 15 ft/sec)
(Parameters given in Table V)
RMW April 1965
RMW

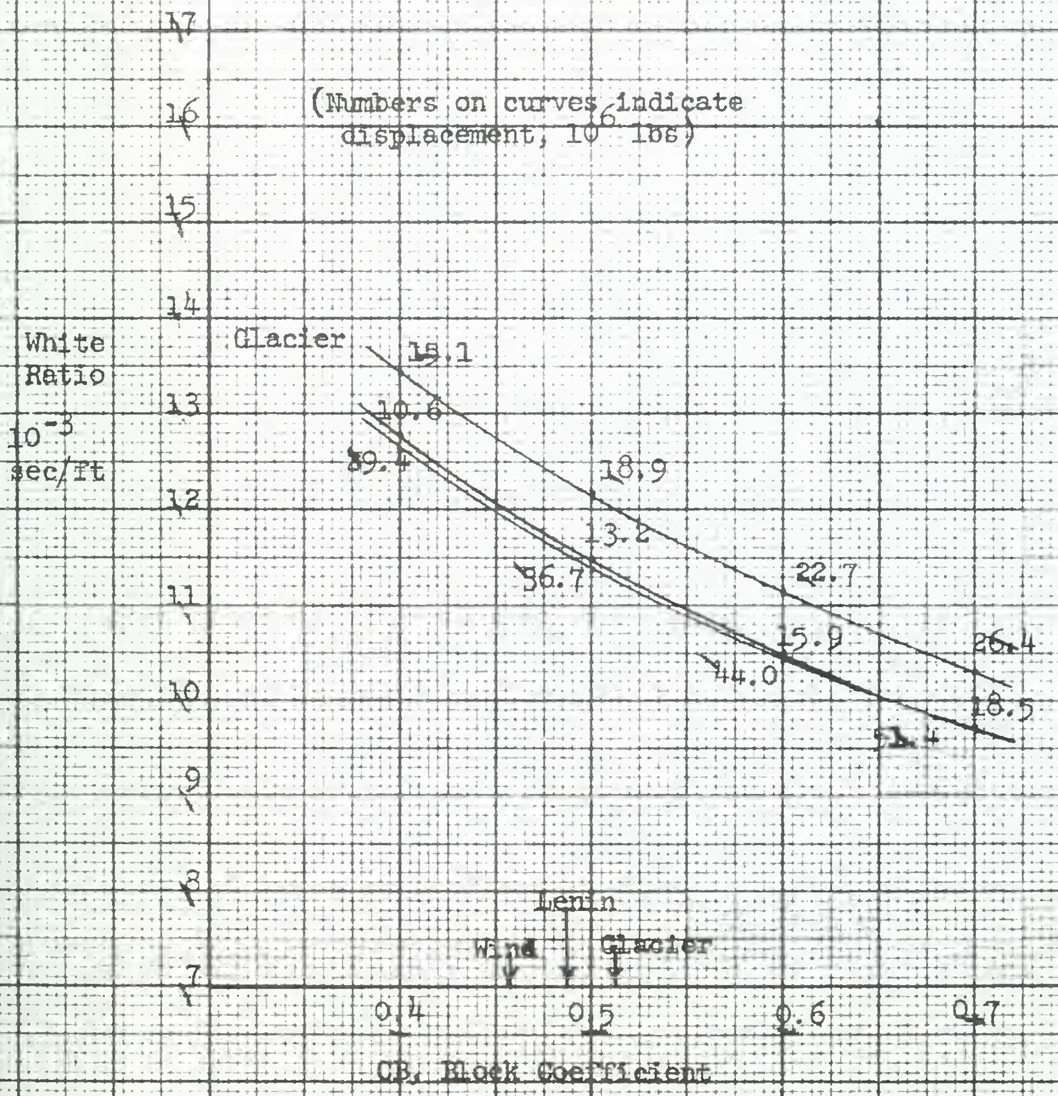


Figure XXXVI

White Ratio

vs

Spread Angle Complement
(Based on VL = 15 ft/sec)
(Parameters given in Table V)
RMW April 1965
RMW

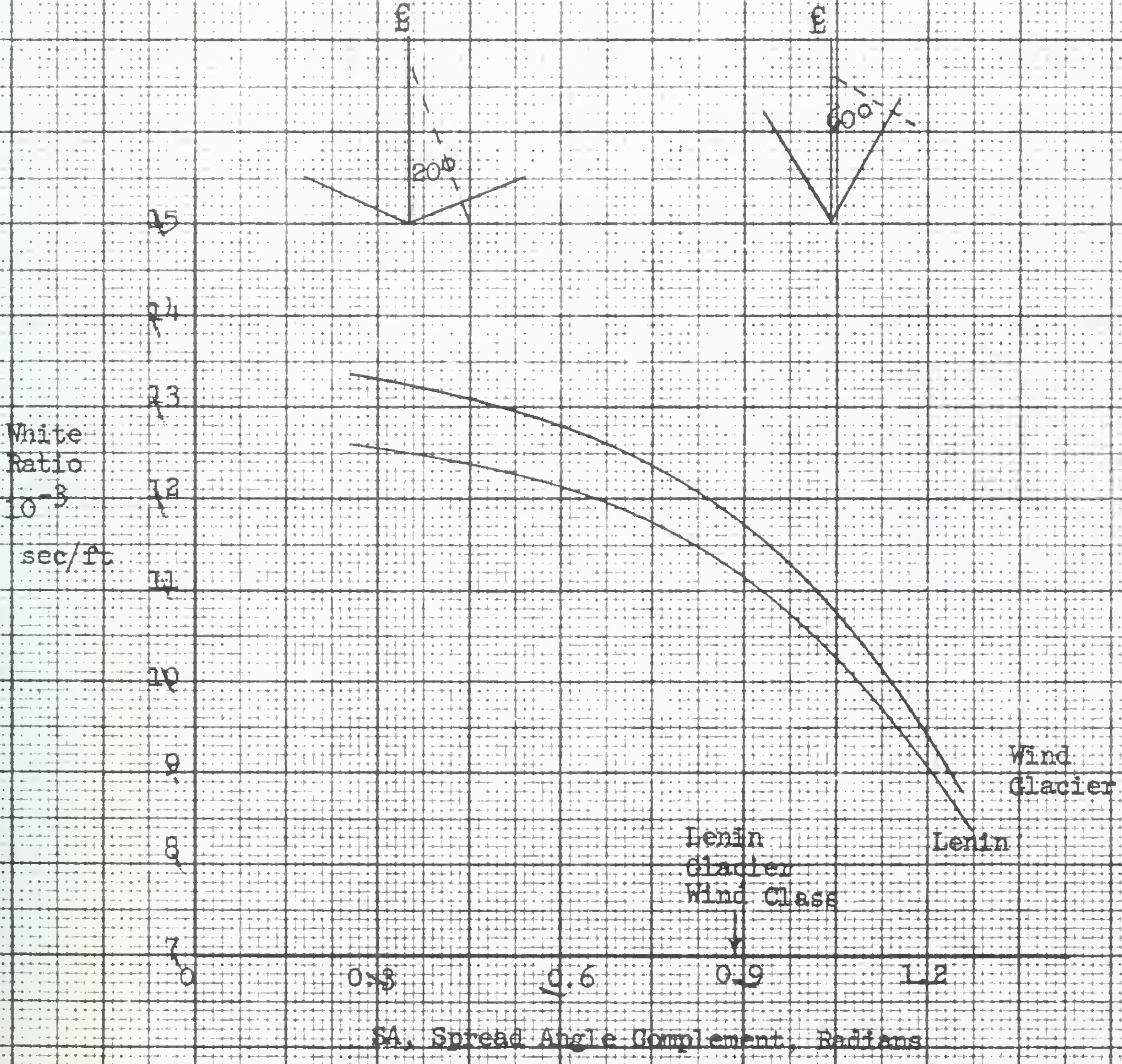


Figure XXXVII
White Ratio
vs
Coefficient of Kinetic Friction
(Based on V1 = 15 ft/sec)
(Parameters in Table V)
RMW April 1965
RMW

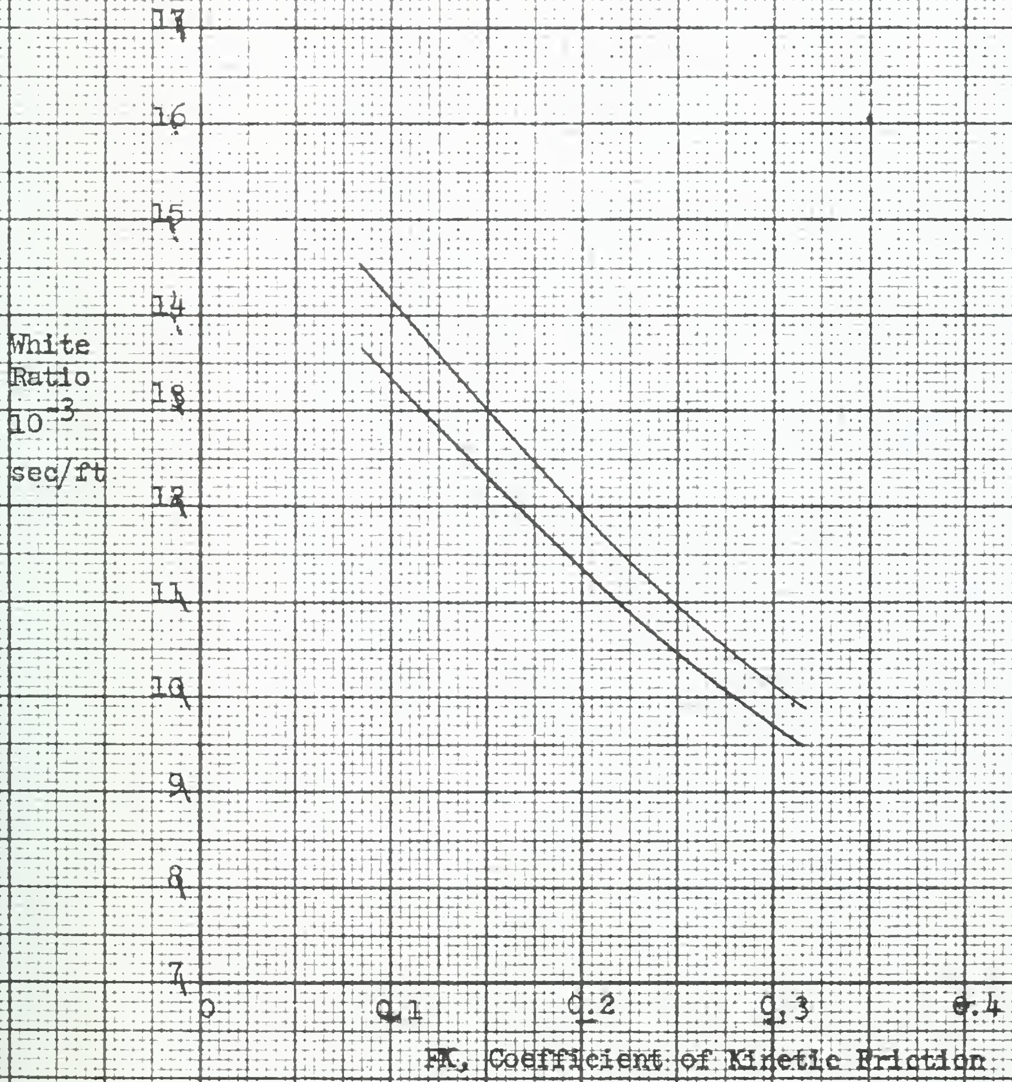


Figure XXXVIII

White Ratio
vs
Bow Angle
(Based on Vl = 15 ft/sec)
(Parameters given in Table V)
RMW April 1965

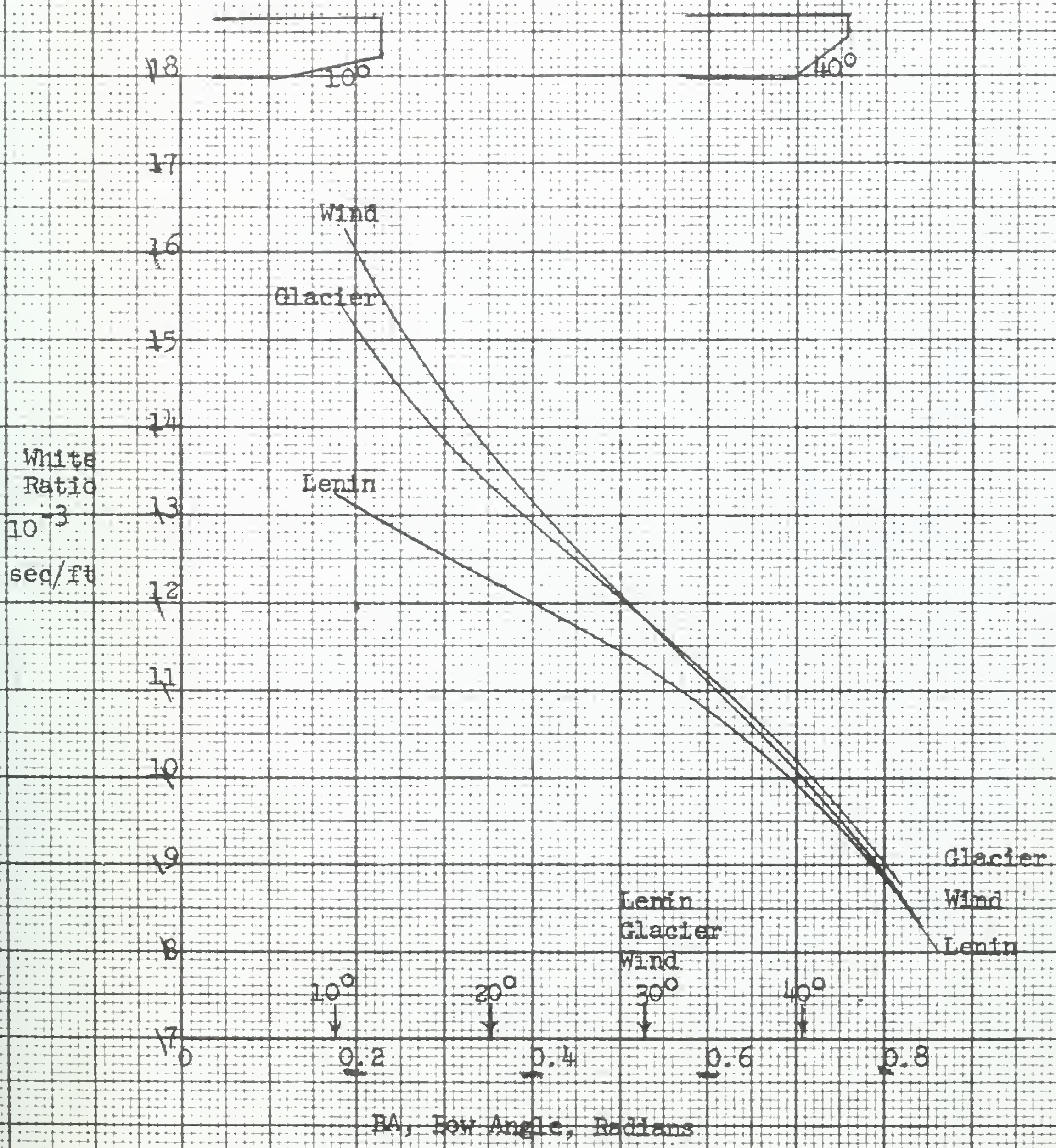


Figure XXXIX
White Ratio for Wind Class
vs
Bow Angle
(Parameters given in Table V)
RMW April 1965

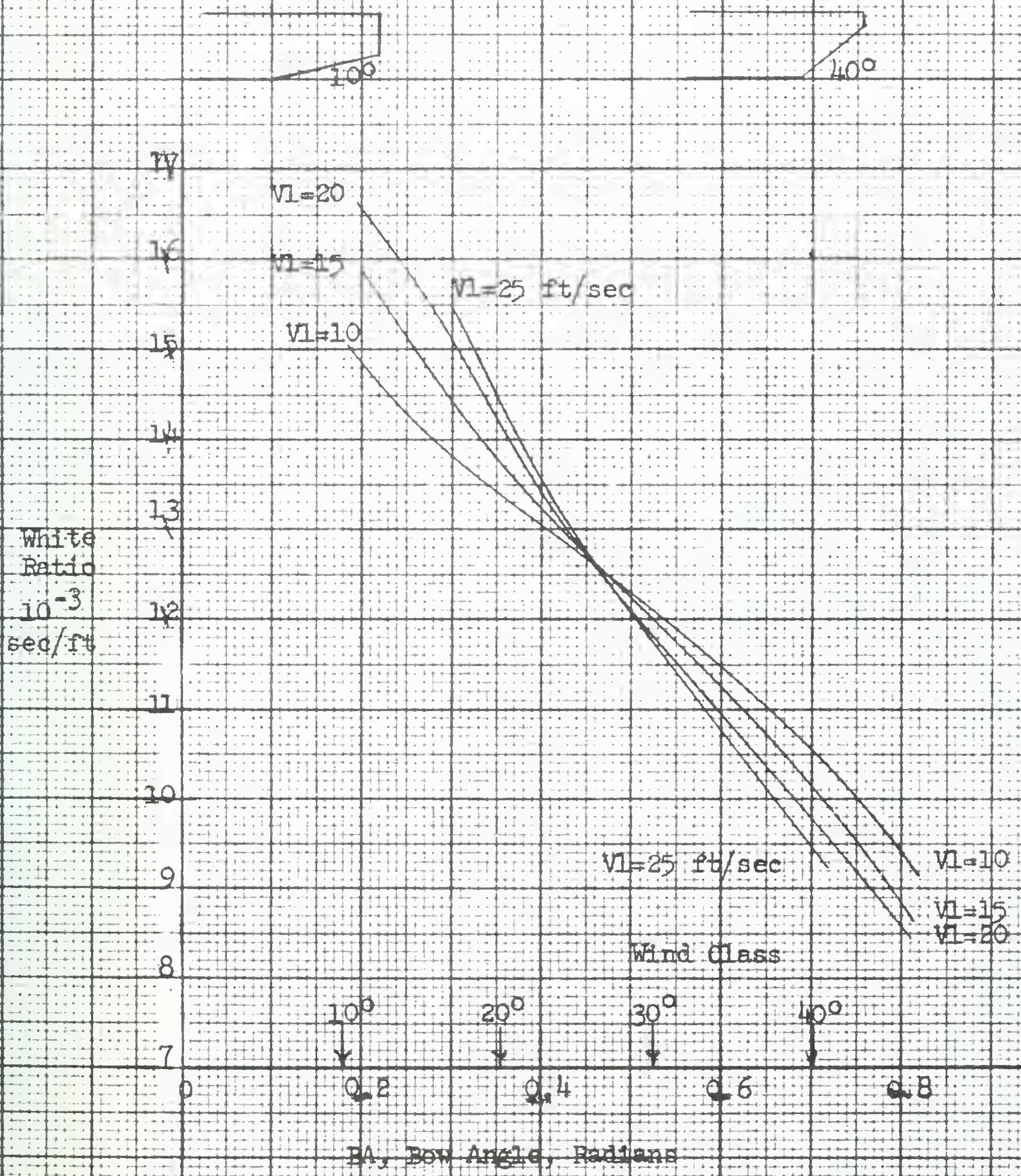
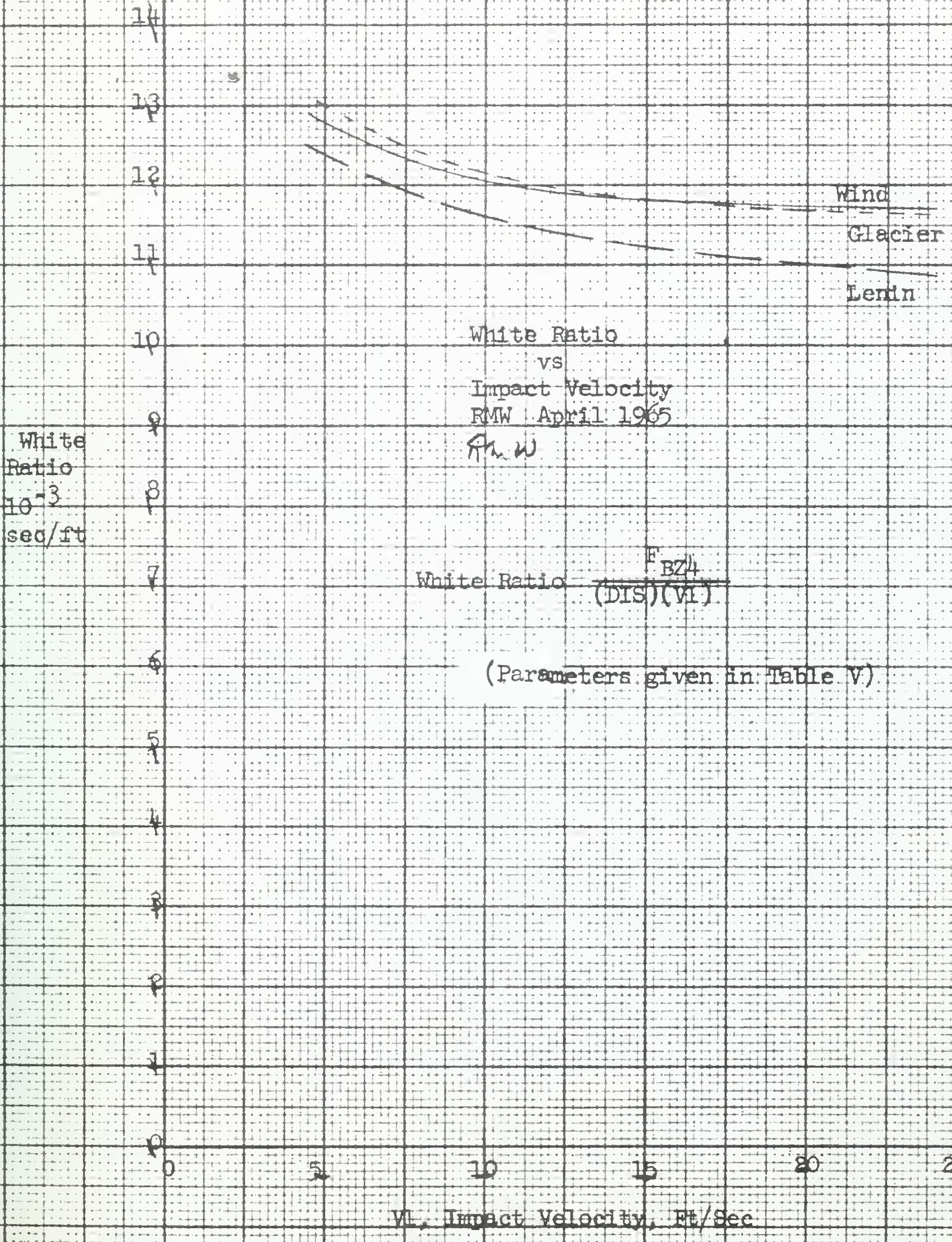


Figure XXXX



White Ratio
vs
Impact Velocity
RMW April 1965

$$\text{White Ratio} \quad \frac{F_{BZ4}}{(\text{DIS})(\text{VI})}$$

(Parameters given in Table V)

Figure XXXXI

Ratio of Extracting Thrust
to Bollard Thrust for Wind Class
vs
Bow Angle
(Based on V1= 15 ft/sec)
(Parameters as given in Table V)
RMW April 1965
RMW

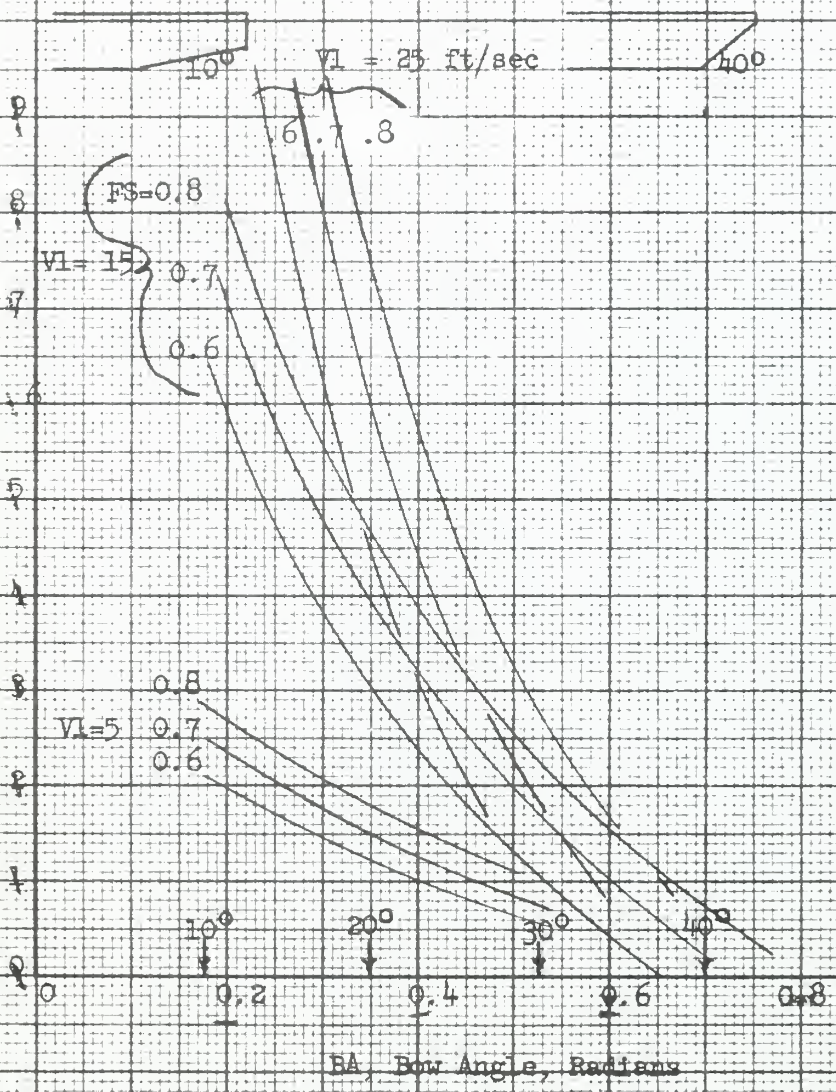


Figure XXXII

Ratio of Extracting Thrust to
Max. Bollard Thrust for Wind Class
vs
Coefficient of Static Friction
(Parameters given in Table V)
RMW April 1965
RMW

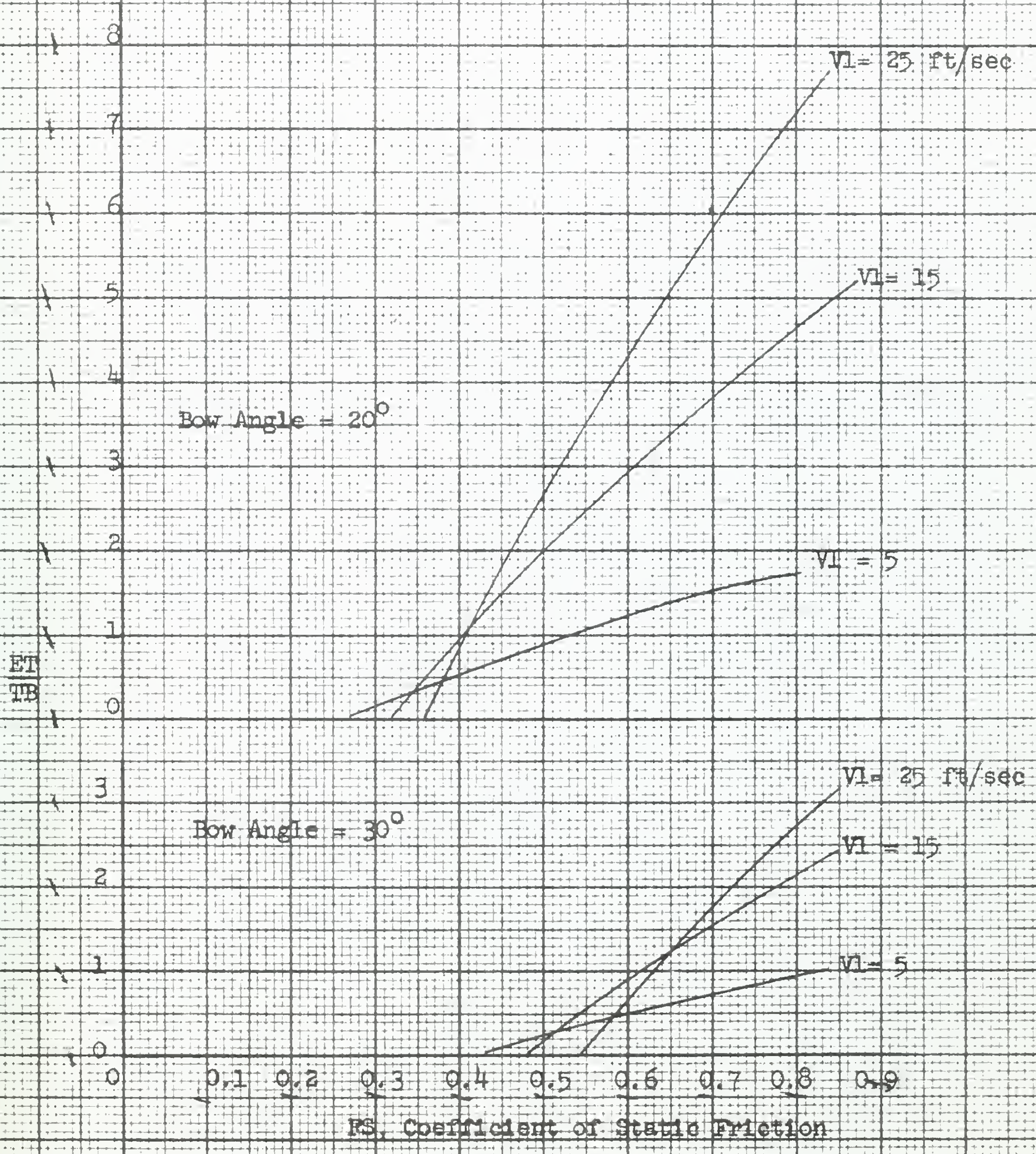
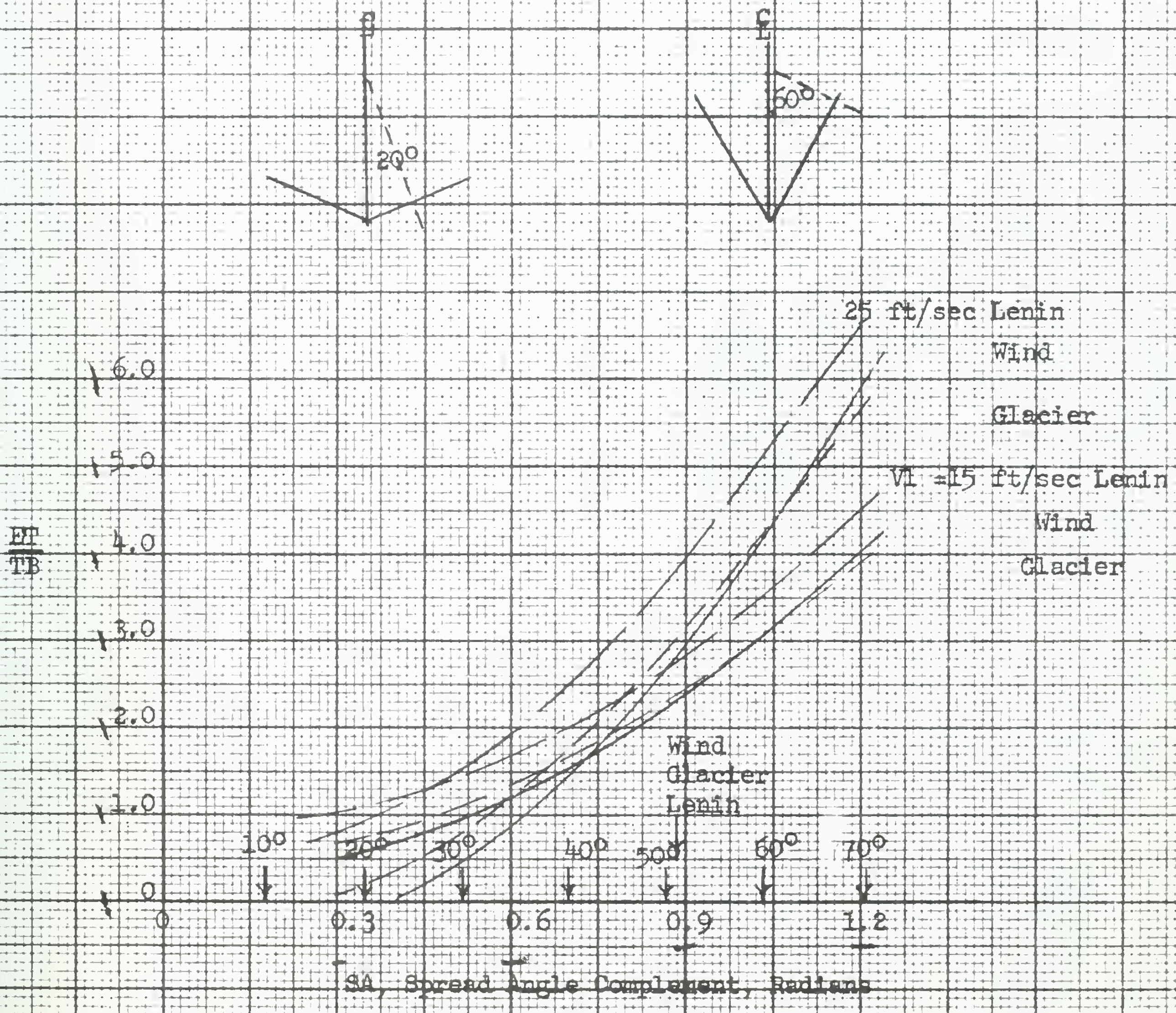


Figure XXXXIII

Ratio of Extracting Thrust
to Bollard Thrust
vs
Spread Angle Complement
(Parameters are given or in Table V)
RMW April 1966
Rn W



Model Correlation

The upper portion of Figure XXXXIV shows the prediction of icebreaking force of a model as a function of time. The lower portion shows the prediction of icebreaking force of a geometrically similar ship as a function of time when the scale ratio, λ, is 100:1. The parameters used in the two solutions are given in Table VI.

Table VII gives the predictions of State 2 (end of crushing) and State 4 (static) values for model and ship respectively.

TABLE VI

PARAMETERS USED IN MODEL-SHIP PREDICTION
(λ = 100) (Wind Class)

Model

BP = 2.500 ft
H = 0.257 ft
BA = 0.523 rad.
Vl = 1.00 ft/sec
CF = -0.012 ft
GK = 0.234 ft
TB = 0.270 ft
FK = 0.20
SIG = 1440.0 lb/ft^2

B = 0.640 ft
DIS = 12.10 lbs
SA = 0.886 rad
AL = 0.724
CG = -0.024 ft
D = 0.16 ft
GM = 1.956 ft
FS = 0.80

Ship

BP = 250.0 ft
H = 25.7 ft
BA = 0.523 rad
Vl = 10.0 ft/sec
CF = -1.25 ft
GK = 23.4 ft
TB = 270,000.0 lb
FK = 0.20
SIG = 144,000.0 lb

B = 64.0 ft
DIS = 12,100,000.0 lbs
SA = 0.886 rad
AL = 0.724
CG = -2.40 ft
D = 16.0 ft
GM = 195.6 ft
FS = 0.80

Figure XXXXIV
Predicted Icebreaking Force of Model and Ship
vs
Time
(Parameters given in Table)

= 100
RMW May 1965

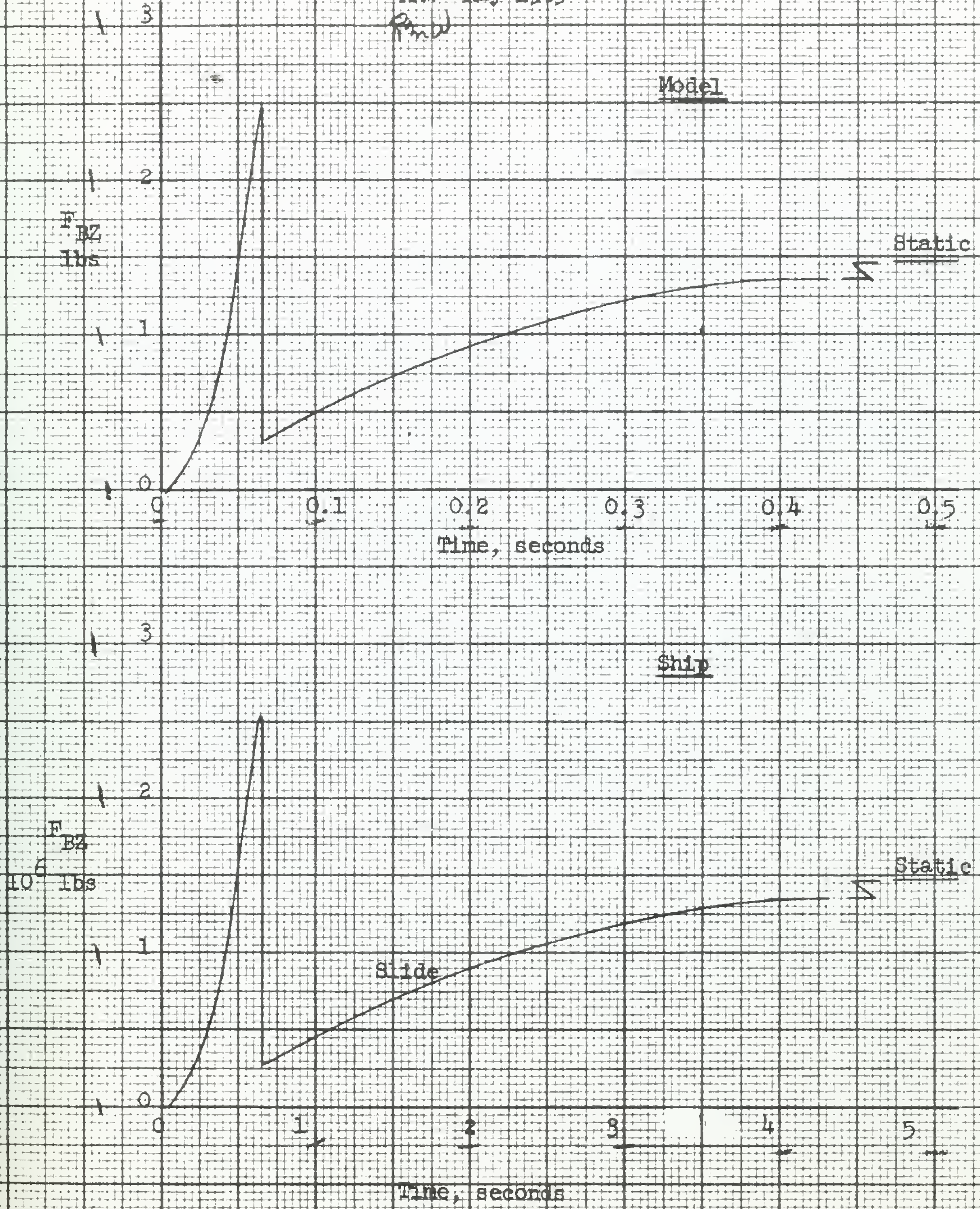


TABLE VII

COMPARISON OF STATES FOR MODEL-SHIP
(λ= 100) (Wind Class)

Model

State 2 $\overset{*}{T}2$ = 0.06298 sec.
TH2 = 0.00555 rad THD2 = 0.33924 THDD2 = 15.35228
Z2 = -0.00111 ft ZD2 = -0.06757 ZDD2 = -3.04988
X2 = 0.06298 ft XD2 = 0.86371 XDD2 = -5.95698
FXC2 = 0.24219 x10'lb FZC2 = 0.24890 x 10' lb

State 4
X4 = 0.24292 ft Z4 = -0.01871 ft TH4 = 0.07567
Vertical force at bow = 0.14587 x 10' lb
White Ratio = 0.120551
Extracting Thrust = 0.46863 lb
Ratio of Extracting Thrust to Bollard Thrust = 1.736

Ship

State 2 T2 = 0.63496
TH2 = 0.00556 rad THD2 = 0.03425 THDD2 = 0.15531
Z2 = -0.11101 ft ZD2 = -0.68218 ZDD2 = -3.08528
X2 = 6.34961 ft ZD2 = 8.63136 ZDD2 = -6.02285
FXC2 = 0.24486 x 10^7 FZC2 = 0.25165 x 10^7

State 4
X4 = 24.35203 ft Z4= -1.87490 ft TH4 = 0.07568
Vertical Force at bow = 0.14591 x 10^7 lb
White Ratio = 0.012058
Extracting Thrust = 0.46874 x 10^6 lb
Ratio of Extracting Thrust to Bollard Thrust = 1.736

IV DISCUSSION OF RESULTS

General

The most significant result of this research is the establishment of a method of solution for determining the downward force under an icebreaker bow; this force is the result of ramming the ice and is a relatively sustained force. The complete computer program is given in Appendix B and yields the force under the bow as a function of 16 inputs. (14 are characteristics of the icebreaker; 2 pertain to the ice.) The object of this research has thus been fulfilled - "a suitable equation for the prediction of the dynamically developed force at the bow of an icebreaker during encounter with virtually unyielding ice".

Validity

Full scale tests were made in 1963 to determine structuralstrains during ramming. As part of these observations other measurements were made (i.e. $\ddot{z}$, θ, $\dot{\theta}$ and $\dot{x}$). Figures XI, XIII, and XV show the predicted and observed values as functions of time. Comparison can be made only up to the time when the bow knuckle comes in contact, as indicated in Chapter II. The agreement of prediction and observation is quite obvious with trim angle, θ. The agreement of prediction and observation is very good with velocity, $\dot{x}$ (with the exception of Run 37B where the observed value was known to be in error).

The correlation between predicted and observed values of acceleration ($\ddot{z}$ and $\ddot{\theta}$) is much better than would seem apparent to a casual observer. It must be recalled that the prediction is for a solid body and that the observations were made on an elastic body. The prediction is essentially an impulse, similar to striking the end of a beam with a sledge hammer. The response (observation) is a vibration of this beam (ship) at its natural frequency. The accelerometers sense and record this vibration and do not feel the impulse that a solid body would have.

Figures VIII, IX, and X show, to a more readable scale, the predictions of x, z, and θ (along with their respective first and second derivatives) for Run 37B. Inspection of these curves reveals a more meaningful representation of the prediction.

Figures XII, XIV, and XVI illustrate one basic idea and observation. The observation shows that there is a peak strain on a forward transverse bulkhead which occurs at about one half second, while the ship doesn't come to a stop until three or four seconds later. This peak is important because it implies that there is a maximum bow force during load crushing. This force may be quite readily seen in the prediction curve for the force under the bow. The time this peak occurs is quite dependent on the compressive failure stress of the ice (although the ultimate value of sustained force is not sensitive to the stress - as will be explained later). For example, using a stress of 144,000 lbs/ft^2 leads to a peak (for Run 37B) at about 0.6 seconds. If 40,000 lbs/ft^2 were used the peak would occur at about 0.9 seconds.

From the above-mentioned comparisons it is realized that the mathematical model of this dynamic motion and the corresponding force under the bow does give a valid representation of real dynamic icebreaking.

Variation of Parameters, Effect on Downward Force*

Compressive Failure Stress of Ice:

As may be seen in Figure XVII the dynamically developed force under the bow is insensitive to the compressive failure stress of the ice. As noted earlier, the impulse peak comes earlier (and is of greater magnitude) when the stress is increased. Obviously the ship designer does not have control of this characteristic so it is indeed fortunate that this parameter is not significant.

Ratio of Height of Thrust Line to Draft:

As used in the calculations, the "height of thrust line" represents the approximate distance from the base line to the shaft line measured at the longitudinal position of the center of gravity.

Figure XVIII illustrates that the downward force is insensitive to the height of the thrust line.

Ratio of Bollard Thrust to Displacement:

It is interesting to note, from Figure XIX, that the application of full power, once initial contact is made, increases the downward force by

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* This section of the discussion is based on Wind Class calculations but is valid for the Glacier and Lenin.

only a few percent. Bollard thrust is very important, but for two reasons not immediately apparent here. High thrust capability is necessary to attain worthwhile impact velocity in a short distance. As will be noted later, backing thrust of large magnitudes is very important.

Ratio of Longitudinal Position of Center of Gravity to Length:

Figure XX illustrates that the downward force is insensitive to the longitudinal position of the center of gravity.

Ratio of Longitudinal Position of Center of Flotation to Length:

Figure XXI illustrates that the downward force is insensitive to the longitudinal position of the center of flotation.

Length-to-Beam Ratio:

Although the beam is important as it affects transverse stability, the width of the channel established, and maneuverability, the length-to-beam ratio has little or no effect on the downward force. This is apparent in Figure XXII.

Ratio of Height of Center of Gravity to Draft:

Figure XXIII indicates that there is a slight advantage in keeping the center of gravity relatively low. There is naturally a gain in transverse stability also. However, this variation should not be considered as significant in the design of an icebreaker since the magnitude of change is only a few percent in a shift of one tenth of the draft.

Beam-to-Draft Ratio:

As may be seen in Figures XXIV and XXXIII, an increase in the beam-to-draft ratio causes a definite increase in the downward force. Although beam and draft are normally determined on the basis of other considerations, where possible a preference should be given to high beam-to-draft ratios.

Waterplane Coefficient:

If the waterplane coefficient is increased (implying a reduction of the waterplane coefficients of immersed waterplanes) there is an increase of magnitude of longitudinal metacentric height. Consequently there is a greater downward force. This may be seen in Figures XXV and XXXIV.

Block Coefficient:

Figure XXVI indicates that the downward force may be increased by increasing the block coefficient. However, the reason for this increase is that the displacement has been increased correspondingly. It is to be noted from Figure XXXV (where the force has been divided by the product of impact velocity and displacement) that increasing the block coefficient decreases the downward force with respect to displacement.

In substance, this means that where a choice is possible, it is preferable to have a large (by linear dimension) icebreaker than a small full one (large C_b) of the same displacement.

Spread Angle Complement:

As may be seen from Figure XXVII, an increase in the spread angle complement (making the bow "sharper") causes a reduction in the downward force which can be attained by ramming. Figure XXXVI illustrates the same result. It is to be noted that there is a significant reduction if bows were to be "sharper" than those on the three major classes investigated.

A decrease of the spread angle complement (making the bow "blunter") causes an increase in the downward force which can be attained by ramming.

As will be discussed later, it is important to note that making the bow "blunter" also decreases the amount of thrust necessary for extraction.

Coefficient of Kinetic Friction:

As would seem obvious, an increase in the coefficient of kinetic friction causes a reduction in the downward force. This may be observed in Figure XXVIII and in Figure XXXVII.

Unfortunately, the coefficient cannot be readily controlled since it depends on the ice as well as the ship. It is apparent, however, that any reduction of this coefficient would be of value. Smoothness of the bow or the application of a durable low friction coating should certainly be considered. (A reduction of 1/10 in coefficient may lead to a 20 o/o increase in downward force.)

Bow Angle:

Probably the most frequently discussed variable of icebreaker design is the angle the stem makes with the base line. As long as the attaining of maximum downward force as a result of ramming is the main consideration, it is of great desire to have a relatively small bow angle. For example, an icebreaker with a 20° bow angle could exert (by ramming) about 20 o/o more downward force than an equivalent icebreaker with a 30° bow angle. This may be observed quite clearly in Figures XXIX, XXXVIII, and XXXXIX.

In Figure XXIX line A-A indicates the condition where the peak load (vertically) during crushing is equal to the final sustained downward force. The area to the right of line A-A is a region where the peak crushing load is greater than the sustained value. For example, at 30° the peak crushing load is about twice the magnitude of the sustained downward force. Therefore it is desirable to reduce the bow angle in order to reduce the relative intensity of this peak load.

Unfortunately, decreasing the bow angle increases the thrust necessary for extraction, as will be explained later.

Displacement:

Figures XXX, XXXI, and XXXII all indicate that an increase in the displacement causes an increase in the downward force, as would be anticipated.

Figure XXX simply shows the effect of displacement by plotting the results of the three major classes of icebreaker investigated. It is significant that the downward force (in the full scale range) is approximately linear with respect to displacement. Figure XXXI is a set of cross curves of the same information.

Figure XXXII shows the effect of increasing displacement by generation of geometrically similar icebreakers. It is clear that the downward force is approximately linear with respect to displacement. It is also clear that "geosims" of the three classes selected produce about the same downward force at any given displacement.

Impact Velocity:

As may be seen in Figure XXXI, the downward force produced as a result of ramming is approximately linear with impact velocity. For example, a Wind Class icebreaker produces a downward force of about 1 1/2 million pounds after impacting at 10 feet per second (about 6 knots); 3 million pounds is produced at 20 feet per second (about 12 knots).

It is also interesting to note that a Wind Class icebreaker can produce, at 15 feet per second (about 9 knots), the same downward that the Glacier produces at 9 feet per second (about 5.3 knots). This is quite significant when one realizes the Glacier has about 60 o/o greater displacement.

Impacting at higher velocities is probably the most productive way of increasing downward force. However, this means that the peak crushing load will be greater also unless the bow angle is reduced from the present practice of 30°.

As will be seen later, the necessary thrust for extraction will probably (but, oddly enough, not "necessarily") be increased.

Higher impact velocities require more thrust for acceleration - and probably more confidence and courage on the part of a commanding officer.

Figure XXXX shows the relative insensitivity of "White Ratio" with variation of impact velocity. Since the "White Ratio" is the downward force divided by the product of impact velocity and displacement, it is again clear that the downward force varies linearly with impact velocity (and displacement).

Extracting Thrust

General:

Since the extracting thrust necessary to pull the ship off the ice is directly related to the downward force under the bow (and the angle at which the static friction is applied), it may be safely stated that practically any variation of parameter which causes an increase in downward force also causes a corresponding increase in extracting thrust.

The effect of change in some parameters is worthy of mention, particularly because there is one notable exception to the above generality.

Static Coefficient of Friction:

As may be readily seen in Figures XXXXI and XXXXII, and as is intuitively obvious, a decrease in the coefficient causes a decrease in the extracting thrust.

The plots use extracting thrust divided by maximum bollard thrust (ahead) as the ordinate. At first glance it would not seem likely that an icebreaker with a 30° bow angle could extract itself if the coefficient of static friction were 0.8 (as used in most calculations). However, in spite of extraction difficulties, all the icebreakers have managed to break free. This is readily explained when one considers the other factors influencing extraction such as shifting the rudder, using trimming and heeling tanks, and explosive charges on the ice.

Experience would indicate that an extracting thrust to bollard thrust ratio of approximately 2 is not unreasonable for a valid icebreaker design (presuming the coefficient is about 0.8). However, experience has shown that we are not far from the threshold with present designs.

Bow Angle:

It is obvious from Figure XXXXI, a decrease in bow angle causes a very significant increase in extracting thrust. For example let us assume that a value of 2 is a tolerable limit for extracting thrust to bollard thrust ratio (as above). Note that we are approximately in that range (or below) with a 30° bow angle. However, if a Wind Class icebreaker had

a 20° bow angle the ET/TB ratio goes to approximately 7 (for 25 feet per second), an obviously intolerable value.

Yet we have seen that, for reasons of increasing downward force and decreasing the relative peak crushing load, it is desirable to decrease the angle. Apparently reduction of the bow angle (below present 30° practice) cannot be wisely undertaken unless there is a reduction of static friction (or, as will be seen later, a "blunting" of the bow).

Figure XXXXII* shows quite clearly that an icebreaker with a 20° bow angle could be operated as safely (from the point of view of extraction) as an equivalent icebreaker with a bow angle of 30° if there were some way of reducing the static friction two tenths (i.e. from 0.7 to 0.5).

It is recommended that strong consideration be given to some method of reduction of static friction. This could be accomplished to some degree by making the bow smoother. It seems probable that durable, low friction coatings could be used. So-called "no stick" coatings are in common use in other applications. They are even used on snow shovels to prevent sticking. Although the use on snow shovels points out the reduction of static friction it does not necessarily represent the

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* The numerical values are also approximately valid for the Glacier and Lenin.

durability. However, the Air Force uses such a coating (Teflon) on the skis of some of its heavy aircraft to prevent adHering to the ice. Investigations along these lines should prove worthwhile.

It is also interesting to note, in Figure XXXXII, that a reversal takes place at low values of static friction. For example, an icebreaker (with a 30° bow angle) requires about half bollard thrust to extract if it has impacted at a low velocity of 5 feet per second (about 3 knots) when the coefficient of static friction is about 0.55. However, little or no backing thrust is required if the impact velocity is 25 feet per second (about 15 knots). This is because it is a somewhat critical region for static friction and the higher impact velocity has given a greater trim angle.

Spread Angle Complement:

As was noted earlier, as an icebreaker bow is made more blunt the downward force is increased. Most significantly however, the necessary extraction thrust is reduced. This may be seen clearly in Figure XXXXIII. For example, if a Wind Class icebreaker with the regular (about 50°) spread angle complement rams the ice at 25 feet per second the extracting thrust is about 3 times the bollard thrust. If the same icebreaker had a blunter bow (about 25° spread angle complement) no thrust at all would be required for extraction.

It is to be noted that this is the only variable which can be

changed and improve downward force and extracting characteristics at the same time. It is recommended that bows for polar icebreakers be designed with a smaller spread angle complement (a blunter bow).

It should be realized that reducing the spread angle complement increases the entrance angle of the bow (measured in the waterplane). However, the entrance angle may be reduced by decreasing the bow angle.

It is quite apparent (from Figure XXXI) that there would be a considerable gain attained in downward force (and relative decrease of peak load during crushing) by using small bow angles. Yet it is also apparent (from Figure XXXXI) that the bow angle, for extraction, should be relatively high. However, the need for this higher bow angle exists only at State 4. Therefore it is recommended that this higher angle exist only at lower sections of the stem, where the stem and bow plating would be in contact with the ice once the forward motion had stopped (State 4).

The result of adopting this idea would be as shown in Figure XXXXV. The stem is slightly concave. The initial contact with the ice would come where the bow is at a 15° to 20° angle. The slope would change continuously down to the lower portion of the stem such that the bow angle would be slightly in excess of 30° in the area which would be in contact during State 4. Particularly considering the recommendation for a small spread angle complement (blunter bow), this should lead to higher sustained downward force, relatively smaller peak load during crushing, and elimination of extraction difficulties.

Model Correlation

As seen in Figure XXXXIV, model predictions may be scaled up to ship predictions based on the equations and scaling factors given in the procedure.

Naturally the model must be geometrically similar. This means that linear dimensions are related by λ and that volumes are related by λ^3. Coefficients remain the same. The compressive failure stress of the ice (or simulating support for the bow of the model) must be related by λ. Model and ship are to be operated at the same Froude Number at impact.

The vertical force at the bow of the ship is λ^3 times the force at the bow of the model. The time-of-ship-event is $\sqrt{\lambda}$ times the time-of-model-event.

The distances and positions are related by λ as may be seen in Table VII. It is noted that extracting forces are related by λ^3.

V CONCLUSIONS

General

The mathematical model of icebreaker motion and corresponding downward force under the bow (given in the Procedure) is valid.

Therefore, the computer program may be used for the prediction of dynamically developed force at the bow of an icebreaker during encounter with virtually unyielding ice.

Effect of Parameter Variation

There is no "optimum" value for any one parameter for maximum downward force. In other words, all curves of downward force as a function of a given parameter are without peak or hollow. (The derivative of the curve does not go to zero.)

The following is a list of causes which will give the effect of increasing the downward force developed by ramming:

Increase of displacement (approximately linear relationship).

Increase of impact velocity (approximately linear relationship).

Decrease of the bow angle.*

Decrease of the spread angle complement (blunter bow).

Decrease of the coefficient of kinetic friction.

Decrease of the block coefficient.

Increase of the waterplane coefficient.

Increase of the beam-to-draft ratio.

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* This decrease of bow angle also lessens the severity of the peak load at impact relative to the final downward force.

The following parameters have little or no effect on the downward force developed by ramming:

Ratio of height of center of gravity to draft.

Length to beam ratio.

Ratio of longitudinal position of center of flotation to length.

Ratio of longitudinal position of center of gravity to length.

Ratio of bollard thrust to displacement (except as explained in Discussion).

Ratio of height of thrust line to draft.

Compressive failure stress of the ice.

Extracting Thrust

With the exception of the spread angle complement, all variations of parameters which cause an increase in downward force also cause an increase in the extraction thrust.

Decreasing the spread angle complement reduces the extracting thrust markedly while improving the downward force characteristic.

It should also be noted that any technique used for reducing kinetic friction (which would increase the downward force) would probably reduce the static friction (which would decrease the extracting thrust).

A reduction of the coefficient of static friction significantly reduces the extracting thrust.

Model Correlation

In icebreaker model tests the results of force may be scaled by λ^3 if the Froude Number of the model at impact is the same as the ship and if the "virtually unyielding ice" of the model test has a compressive failure stress equal to the failure stress of the ice divided by λ.

The position may be scaled using λ based on the relationship that the time-of-ship-event equals $\sqrt{\lambda}$ times the time-of-model-event.

VI RECOMMENDATIONS

General

It is recommended that the characteristics of any proposed polar icebreaker be used in the mathematical model (computer program) to investigate the downward force developed by ramming. The program should also be used to determine extracting thrust and the peak load of the crushing phase.

Selection of Characteristics

If attainment of the maximum downward force were the prime objective in the design of a polar icebreaker, the following choices would be significant:

Large displacement
High impact velocity
Small bow angle
Small spread angle complement
Low value of kinetic friction (dependent in part on the ice).
Small block coefficient (if displacement is constant)
Large waterplane coefficient
High beam-to-draft ratio

The following characteristics may be disregarded (concerning downward force):

Ratios of
Height of center of gravity to draft
Length to beam

Longitudinal position of center of flotation to length

Longitudinal position of center of gravity to length

Bollard thrust to displacement (except as explained in Discussion)

Height of thrust line to draft

The free selection of apparently desirable characteristics is limited by the extracting requirement. It is recommended that extracting thrust requirements be kept in mind (and evaluated) when selecting characteristics.

Decreasing the spread angle complement and reducing friction in general (both static and kinetic) are the only ways of simultaneously increasing downward force and reducing the extracting thrust requirement.

It is recommended that future polar icebreakers have blunter bows (measured in a plane perpendicular to the stem).

It is furthermore recommended that significant efforts be made to reduce friction between the hull and the ice, particularly static friction. One of the most hopeful solutions is in the use of durable "no-stick" coatings as discussed previously. Other techniques may also be possible (i.e. "lubrication" or heating).

If a useful technique for reducing this static friction becomes possible, then it would be recommended that the bow angle be selected from values less than the presently used 30°.

Thrust should be as great as possible commensurate with other considerations. A larger thrust allows higher rates of steady icebreaking. A large thrust allows greater acceleration in a relatively shorter

distance to attain a desired impact velocity, which is quite important. Backing thrust is equally important. If higher backing thrusts were available then it would be possible to select characteristics which would increase the downward force under the bow without as severe a limitation imposed by extracting requirements.

Since high thrust at low speed (or zero speed) is extremely desirable, efforts should be made to select (or design) for maximum backing thrust at 100 percent slip, even at the sacrifice of open water efficiency. Although some work has already been done (80) concerning the use of Kort Nozzles, much more investigation is needed and recommended.

Model Testing

It is recommended the model test of ramming be undertaken using a Froude Number for model operation which is the same as the ship at impact.

The model that may be similar to that used by Richardson (16) and by McMahan and Abrahams (40). However, it is necessary that the material used as "ice" have a significantly lower compressive failure stress specifically $1/\lambda$ times the compressive failure stress of ice. This will allow local crushing to accommodate the bow of the model to the same relative degree as the bow of the ship. This will lead to results which may be scaled.

Care must be taken to insure that vibration of the support for the

"ice" is eliminated or minimized.

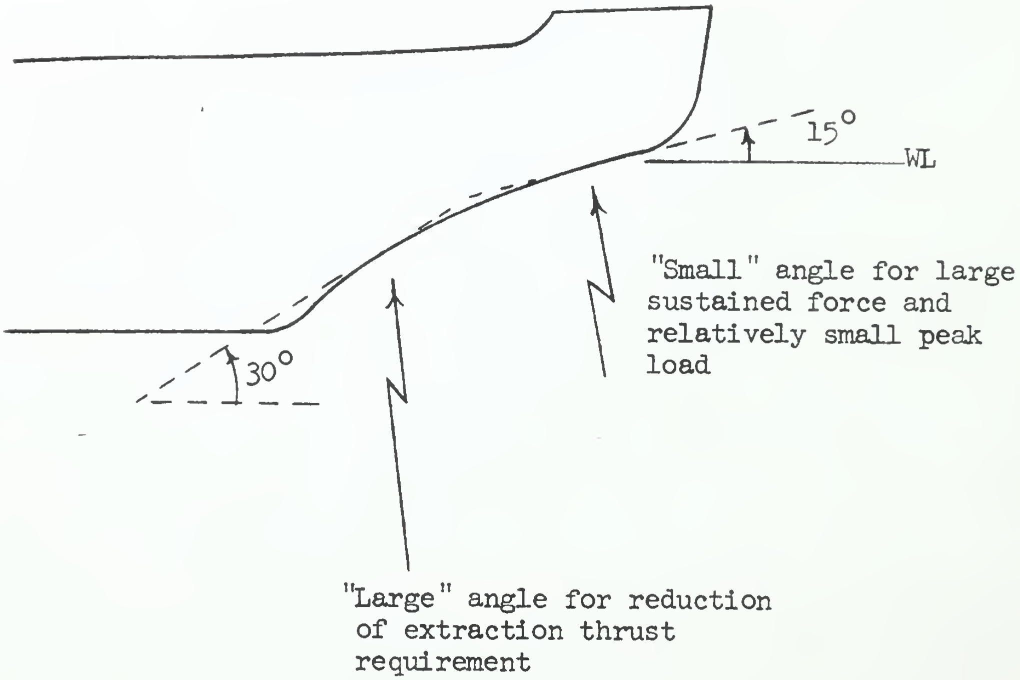
Through such tests other effects, such as loss of stability when encountering virtually unyielding ice, may be examined.

Bow Shape

It is recommended that the bow shape of polar icebreakers incorporate the ideas illustrated in Figure XXXXV. The angle at initial entry should be small (i.e. 15° to 20°) and the stem should be concave such that the area in contact with the ice after stopping have a relatively steeper slope (i.e. 30° - 35°). The spread angle complement should be relatively higher (blunter), perhaps in the order of 0.6 radians (34°). Compared to present bow shapes, this recommended shape will lead to greater downward sustained force, relatively smaller peak load during crushing, and elimination of extraction difficulties.

Figure XXXXV

Recommended Polar Icebreaker
Bow Profile


15°
WL
"Small" angle for large sustained force and relatively small peak load
30°
"Large" angle for reduction of extraction thrust requirement

VII APPENDIX

[illegible]

A. SUPPLEMENTARY INTRODUCTION

Runeberg's Equation

In 1888 an equation was published (10) for the determination of the vertical component of force produced at the bow of an icebreaker during uninterrupted progression.

Runeberg* used the following symbols for his development:

V = "Vertical pressure at bow", Vertical component of force at bow in lb.

R = Thrust of the propeller in lb.

θ = Trim in deg. (change of trim)

Δ = Displacement in t.'

ϕ = Angle of inclination of buttock lines to the waterline, (stem angle) in deg. See Figure A-I.

b = "Inclination of cross sections taken perpendicular to the buttock lines with respect to the waterline". (His use of this term indicates that it is the complement of the angle from the ℄ plane to the hull measured in a plane which is perpendicular to the stem.) See Figure A-II. Expressed in deg.

v = Velocity in ft/sec.

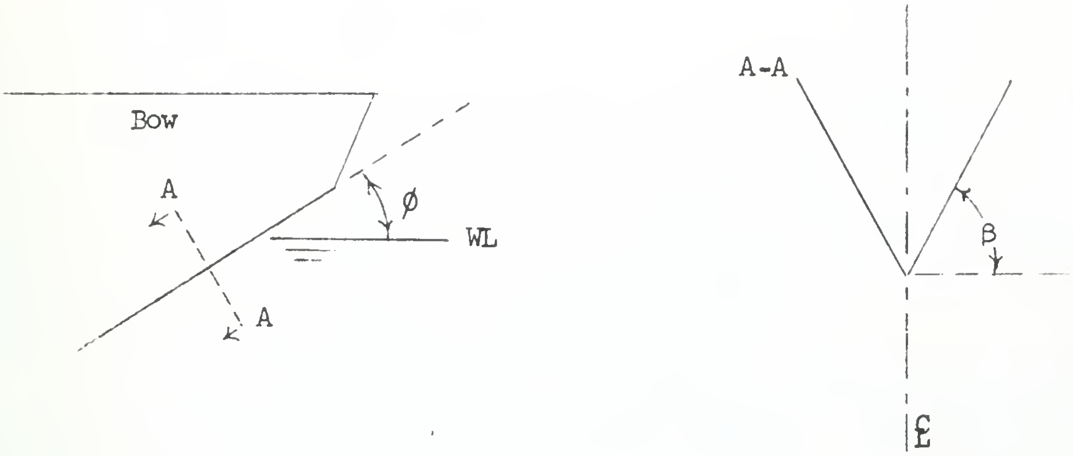
δ = Mean decrease in draft in ft.

Q = "Pressure normal to buttocks". (His use of this term indicates it is the force in the ℄ plane normal to the stem.) Expressed in lb.

N = Total force perpendicular to the bow plating. (Note that $N/2$ acts on each side of bow.) Expressed in lb.

f = Coefficient of friction of bow plating relative to ice while moving.

Figure A-I
Illustration of Terms Used by Runeberg



P = Propeller pitch in ft.

No = Shaft r p m .

p = "Mean effective pressure on total piston area". (His use of this term indicates that it is the mean effective pressure multiplied by the total area of the pistons.) Expressed in lb.

S = Length of stroke in ft.

Runeberg developed his equation for the vertical component of force at the bow using the equilibrium equation based on Figure A-II. It is to be noted that his figure does not agree with presently accepted standards of notation but still leads to an acceptable result.

V was drawn perpendicular to the waterline. AB represents the line of the stem and the buttocks in the area of contact. Q was drawn perpendicular to AB.

The ship slides up (neglecting momentum) to a point where the force downward along the stem becomes equal to the force pushing the bow upward along the line indicated by the slope of the stem. At that point

$$K = R \cos \phi \qquad (A1)$$

$$\text{where } K = V \sin \phi + f\,N \qquad (A2)$$

It follows that

$$R \cos \phi = V \sin \phi + f\,N \qquad (A3)$$

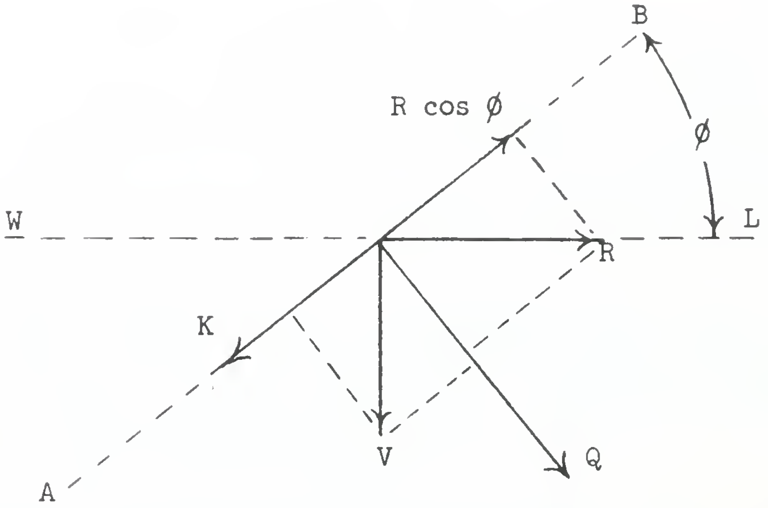
This can be put into the following form:

$$V = \frac{R \cos \phi - f\,N}{\sin \phi} \qquad (A4)$$

He indicates that the thrust provided (by pressure on pistons) is divided into six parts according to Froude.

Figure A-II

Bow Equilibrium by Runeberg



1. Useful thrust (normally equal to the ship's own net resistance).
2. Augmented resistance due to action of propeller.
3. Friction of screw blades against water.
4. Slow speed friction of engine.
5. Working load friction of engine.
6. Resistance due to air and feed pumps, etc.

If the ship is pushing against the ice the last five remain unchanged but the ship's own resistance is equal to zero and in its place is the thrust R. (He has assumed no advance through the ice and that all useful thrust can be used against the ice.)

Runeberg assumes that 37.5 o/o of indicated thrust is that portion which goes to "the ship's own net resistance". Therefore he simply transfers this amount to use for ice-breaking.

$$R = \frac{37.5}{100} \times \frac{\text{IHP} \times 33{,}000}{P \times \text{No}} = \frac{12{,}375\ \text{IHP}}{P \times \text{No}} \qquad \text{(A5)}$$

where $\text{IHP} = \frac{2\ p\ S\ \text{No}}{33{,}000}$

Rewriting equation (A5),

$$R = \frac{0.75\ p\ S}{P} \qquad \text{(A6)}$$

As can be seen from Figure A-III,

$$Q = (2)\left(\frac{N}{2}\right)\cos\beta = N\cos\beta \qquad \text{(A7)}$$

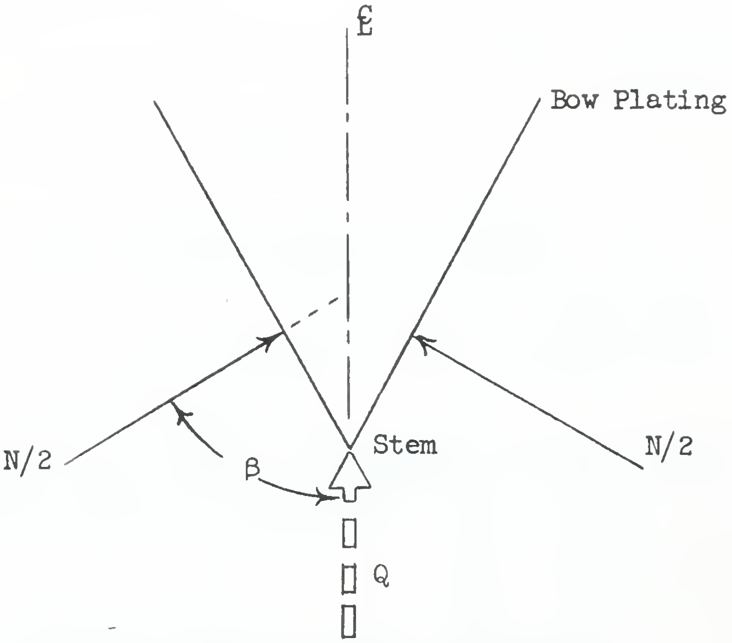
where Q is in the plane and perpendicular to the stem.

Again referring to Figure A-III, it can be seen that

$$Q = R\sin\phi + V\cos\phi \qquad \text{(A8)}$$

Figure A-III

Resolution of Forces Normal to Bow Plating



(looking down line at stem)

Substituting for Q this becomes

$$N \cos \beta = R \sin \phi + V \cos \phi$$

or

$$N = \frac{R \sin \phi + V \cos \phi}{\cos \beta} \qquad \text{(A9)}$$

By substituting equations (A6) and (A9) into equation (A4) the following equation results:

$$V = \frac{0.75\ p\ S \cos \phi}{P \sin \phi} - \frac{0.75\ f\ p\ S \sin \phi}{P \sin \phi \cos \beta} - \frac{f\ V \cos \phi}{\sin \phi \cos \beta} \qquad \text{(A10)}$$

Then this can be rewritten to

$$V = \frac{12{,}375\ \text{IHP}\ (\cos \phi \cos \beta - f \sin \phi)}{\text{No}\ P\ (\sin \phi \cos \beta + f \cos \phi)} \qquad \text{(A11)}$$

Converting the symbols used in equation to those used generally in this research, the equation becomes

$$F_{BZ} = \frac{T_{IB}\ (\cos i_B \cos \beta - f_k \sin i_B)}{(\sin i_B \cos \beta + f_k \cos i_B)} \qquad \text{(A12)}$$

or

$$F_{BZ} = \frac{12{,}375\ (\text{ihp})\ (\cos i_B \cos \beta - f_k \sin i_B)}{P\ (\text{rpm})\ (\sin i_B \cos \beta + f_k \cos i_B)} \qquad \text{(A13)}$$

where Runeberg suggests that $f_k = 0.05$.

The following assumptions were made during this development:

1. There are no momentum effects.
2. The forward motion through the water is effectively non-existent so that the thrust can therefore all be applied to icebreaking*.

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* These assumptions were not stated.

3. Thrust was directed horizontally at all times*.
4. The direction of friction force (along the line of the stem) remains the same during forward horizontal progress*.

His equation was developed on the basis of a ship sliding up on the ice very slowly* under the influence of its own thrust. It is deliberately approximate and was developed to be used for uninterrupted progress.

Runeberg does go on to develop some ideas concerning the "icebreaking power of a steamer when charging" but of these there is no direct connection to the forces developed at the bow.

If a ship is charging the ice he indicates that it will have "momentum" equal to $\frac{D\,v^2}{2g}$, where D = displacement in pounds. This "momentum" (actually kinetic energy) will be employed in the following two ways:

1. Elevating the ship
2. Overcoming frictional resistance as the bow glides up on the ice.

Later he mentions the work added by means of thrust while in ice contact. He indicates that there is an increase of frictional resistance due to an increase of normal pressure which is brought about "by the center of gravity of the ship changing direction of motion after the bow has struck the ice". Although he does not use it to advantage, this is the only mention of this particular dynamic force to appear up to this date (1964).

Unfortunately on the other hand, he presumes the loss by concussion is insignificant.

His concern over ramming does not lead to any prediction of force at the bow.

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* These assumptions were not stated.

Kari's Equation

In 1921 a book entitled "The Design of Icebreakers" by A. Kari was published (11). An equation is developed which does give the downward component under the bow during icebreaking. Kari developed this in order to determine certain characteristics the vessel should assume in order to break a given thickness of ice. As written the development leaves much to be desired. It is paraphrased and clarified here somewhat.

The following symbols are used:

W = Displacement, tons

R = Upward ice resistance, tons

θ = Inclination of stem to horizontal (original), deg.

ϕ = Maximum permissible angular displacement of LWL, deg.

$\pm a$ = Distance of the center of flotation forward (+) or aft (-) of amidships, ft.

L = Length of LWL, ft.

D = Moulded mean draft, ft.

GM' = Longitudinal metacentric height, ft.

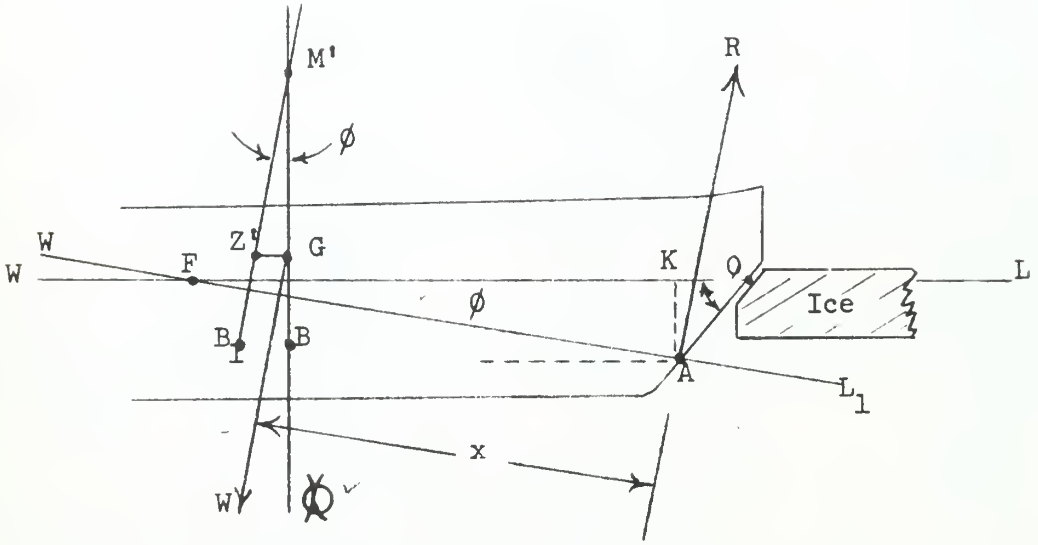
t = Maximum thickness of ice to be expected, ft.

Figure A-IV illustrates many of these symbols.

Consider the locus of the point of initial contact; it moves along a somewhat circular path. Kari states, "This is the result of angular oscillation about the center of gravity and the gradually reducing forward motion. A force is produced by the angular displacement of the ship's waterline. The center of buoyancy is shifted aft and a trim moment is provided which, being divided by the separation of the point of contact with the ice from the center of gravity, provides the breaking power".

Figure A-IV

Illustration of Symbols Used by Kari



This statement becomes reasonable if one substitutes "center of flotation" for "center of gravity".

The trimming moment at a trim ϕ can be expressed as

$$W \times G\,Z' = W \times G\,M' \sin \phi \qquad (A14)$$

Referring to Figure A-IV it can be seen that

$$AK = AF \sin \phi \qquad (A15)$$

Without stating the equality or the reason for it, Kari then sets $AK = t$. It is important to note that, in order to continue with any logic, it is necessary to redefine t.

t = Rise of point A in ft.

Then

$$t = AF \sin \phi \qquad (A16)$$

Kari states that AF is approximately equal to x. Therefore

$$t = x \sin \phi \qquad (A17)$$

Summing moments about the center of gravity and setting them equal to zero he gets

$$W \times G\,M' \times \sin \phi = Rx \qquad (A18)$$

Using equation (A17) x may be eliminated.

$$R \times \frac{t}{\sin \phi} = W \times G\,M' \times \sin \phi \qquad (A19)$$

or

$$R = \frac{W \times G\,M' \times \sin^2 \phi}{t} \qquad (A20)$$

Assuming that G M' is approximately B M' and that $B\,M' = \frac{CL^2}{D}$ one gets

$$R = \frac{W x C x L^2 \times \sin^2 \phi}{D^x t} \qquad (A21)$$

where $C = 0.07$.

From this point he goes on to end up with a rather astonishing result which will not be developed here. The equation gives the necessary length of a ship as a function of bow angle, trim, location of the center of flotation, and the ice thickness.

$$L = 2\ t\ (\cot \theta + \cot \phi) \mp 2\ a \qquad \text{(A22)}$$

where t in this case reverts to the original definition of "maximum thickness of ice to be expected, ft."

Returning now to the downward force, R, as seen in equations (A20) and (A21), it is necessary for one to use an approximation for the vertical rise of point A which is indicated by "t" in the equations. It will be no less logical than many of the assumptions he has used to substitute L/2 sin ϕ for t in order to obtain a more useful form. Equation (A20) becomes

$$R = \frac{W \times G\ M' \times \sin \phi}{L/2} \qquad \text{(A23)}$$

and equation (A21) becomes

$$R = \frac{W \times C \times L \times \sin \phi}{D/2} \qquad \text{(A24)}$$

where C = 0.07.

Converting the symbols used in equations (A23) and (A24) to those used generally in this research, the equations become

$$F_{BZ} = \frac{2\ \Delta\ GM_L\ \sin \theta}{L} \qquad \text{(A25)}$$

and

$$F_{BZ} = \frac{2\ \Delta\ C\ L\ \sin \theta}{H} \qquad \text{(A26)}$$

where C = 0.07.

The equations for the downward component do not indicate the maximum and are approximately valid only for a motionless case.

In arriving at equations (A20) and (A21) Kari used the following assumptions or expedients:

1. There are no momentum effects.
2. The vertical rise of the bow is equal to the thickness of the ice. (The effect of this assumption was nullified by redefining the symbol t.)*
3. The distance from the final point of contact with the ice to the center of flotation is assumed to be the same as the horizontal distance from the final point of contact to the center of gravity.
4. The effective displacement is not effected by the force at the bow, nor is the draft*.
5. Following 4, the center of flotation and the longitudinal metacenter remain fixed*.
6. The normal assumption is made that $GM_L = B\,M_L$.
7. The value of C in equation (A21) is 0.07.
8. Friction is neglected*.
9. It is insignificant but G Z' should be shown perpendicular in Figure A-IV.

---

* These assumptions were made but not stated.

10. The summation of moments was set equal to zero. If forces had been summed there would have been a discrepancy.

In summary, Kari's equation states that the vertical component of force, F_{BZ}, in tons is a function of the following:

Δ, displacement, tons

GM_L, longitudinal metacentric height, ft.

θ, change of trim, deg.

L, length between perpendiculars, ft.

or

H, draft, ft. (instead of GM_L)

Simonson's Equations

In a paper published in 1936 (12) an equation was presented giving the force available for breaking ice. This is the steady state vertical reaction which results when the vessel is forced out of her normal waterplane by the thrust of the propeller.

The following symbols were used in this development:

W = Vertical reaction at the bow at the point of contact with the ice in lb.

M = Trimming moment of the vessel to change trimming 1 in. expressed ft-t/in.

T = Thrust of the propeller in lb.

Y = Allowable trim in in.

D = Distance from the center of flotation to portion of the stem in contact with the ice in ft.

L = Length between perpendiculars in ft.

Δ = Displacement in t.

GM = Longitudinal metacentric height in ft.

ϕ = Change in trim in deg.

K = Velocity expected through the ice in kt.

HP = Total horsepower available less the amount necessary to drive the ship at speed K (in open water), hp.

f = Overall efficiency of power plant and propeller at speed K when developing maximum horsepower; varies between 10 and 25 percent.

C_t = Thrust coefficient, (These units must be lb-sec^2/ft^4.)

N = Revolutions per second obtainable at speed K at rated horsepower, rps.

P = Propeller pitch in ft.

d = Propeller diameter in ft.

Θ = Angle between the stem and the surface of the ice field in deg.

γ = Angle between stem and waterplane (original) in deg. (This is the designed stem angle.)

The vertical reaction, W, is due to the trimming moment, M, when the vessel is forced out of normal waterplane by the thrust, T. This is expressed in the following approximate equation:

$$W = \frac{2240\ M\ Y}{D} \tag{A27}$$

The moment to change trim one inch can be expressed as

$$M = \frac{\Delta\ G\ M}{12\ L} \tag{A28}$$

Simonson assumes that G M can be approximated by L. Then

$$M = \frac{\Delta}{12} \tag{A29}$$

He furthermore assumes that when the bow is not cut away too much and when the trim is small (less than 5°), D can be approximated by L/2.

By substitution in equation (A28) he gets

$$W = \frac{2240\ \Delta\ 12L\ \tan \phi}{12\ L/2} \tag{A30}$$

$$W = 4480\ \Delta\ \tan \phi \tag{A31}$$

Converting the symbols used in this equation to those used generally, the equation becomes

$$F_{BZ} = 4480 \Delta \tan \theta \qquad (A32)$$

Note, this equation simply states that if the displacement and the trim (caused by pushing the bow up on ice) are known, the vertical component of the force at the bow can be solved.

Remember F_{BZ} is defined as the force against the bow. Therefore it is positive. Naturally the magnitude downward against the ice is the same.

In arriving at this equation it should be noted that Simonson made the following assumptions:

1. G M (longitudinal) = L
2. D = L/2
3. M remains constant*
4. Displacement remains constant*.
5. Longitudinal metacentric height remains constant.*

It is to be emphasized that equation (A32) is intended for steady state icebreaking. However, by itself it does not give the maximum force.

In order to find a maximum it is necessary to determine thrust. The method he uses employs the following equation:

$$T = \frac{\text{HP x f x 33,000 ft.lb./min. x 60 min/hr}}{\text{K x 6080 ft/hr.}} \qquad (A33)$$

---

* These assumptions were not stated.

$$T = \frac{325.7 \times HP \times f}{K} \text{ lb.} \qquad (A34)$$

Other approximations for thrust are as follows:

$$T = 22.40 \times IHP \qquad (A35)$$

(one ton thrust per 100 IHP)

or

$$T = C_x N^2 P^2 d^2 \qquad (A36)$$

The angle γ (stem angle) is represented by Figures A-V and A-VI. In the case of the curved bow shown in Figure A-V, the angles are expressed by tangents to the stem profile.

He assumed that friction in the steady state was negligible although this is not stated. His solution of the equilibrium was based on forces and he assumed that thrust remained horizontal. Since the summation of moments was not introduced, it is irrelevant that he did not mention the line of action of thrust relative to the point of contact.

From Figure A-VI it can be seen that

$$\gamma = (\theta - \phi) \qquad (A37)$$

and that

$$\tan \theta = \frac{T}{W} \qquad (A38)$$

From this point he goes on to substitute equations (A31), (A34) and (A38) into equation (A37) to get

$$\gamma = \tan^{-1} \left(\frac{0.0727 \times HP \times f \cot \phi}{\Delta K} \right) - \phi \qquad (A39)$$

Figure A-V (12)
Illustration of Symbols Used by Simonson

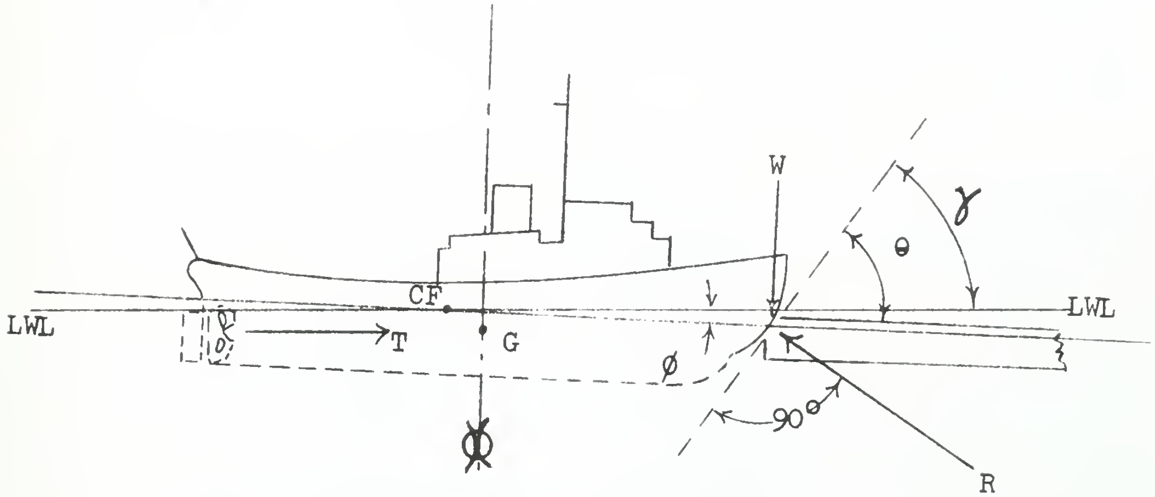
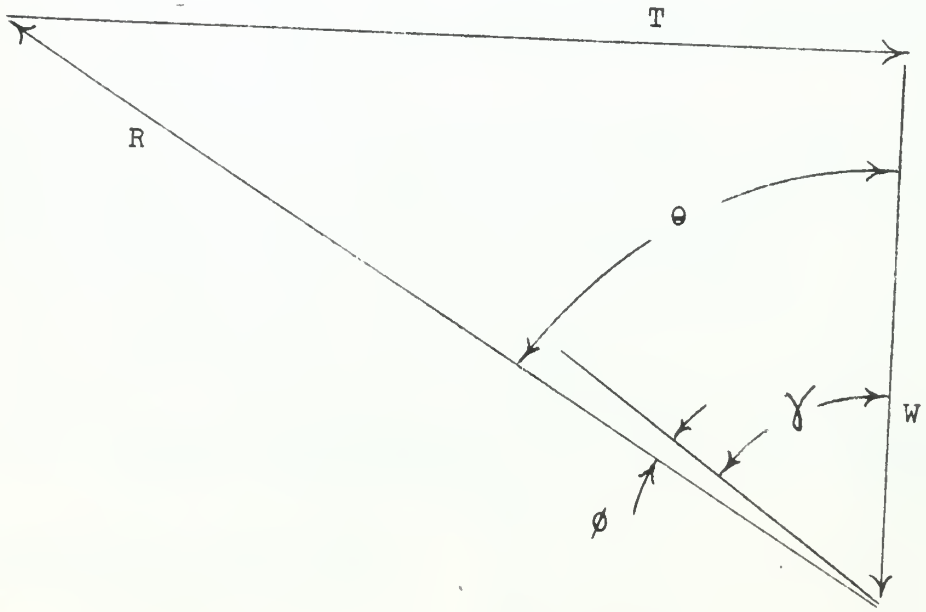


Figure A-VI (12)

Bow Equilibrium by Simonson



γ then simply indicates the stem angle which should exist if T, R, and W are to be in equilibrium. (R is the force perpendicular to the stem.) He uses this to obtain a bow profile which would represent an equilibrium condition regardless of the trim if other factors (i.e. thrust) were held constant.

Incidentally, this equation, although it is of no direct significance concerning this research, is given as follows in order to demonstrate Simonson's goal:

$$X = \left(6\ L + \frac{0.436\ HP \times f \times L}{\Delta\ K}\right) .$$

$$\log\left(\frac{10.5\ L^2 \times HP \times f}{\Delta\ K} - Y^2\right) \qquad \text{(A40)}$$

where $\cot \phi = \frac{12\ L}{Y}$ and $\cot \theta = \frac{W}{T}$

Y = trim in in.

X = distance in inches from center of flotation to stem on waterplane of trim.

Utilizing his equations it is possible to deduce an equation for maximum (limiting) downward force available from a given ship.

Starting with equation (A38)

$$\tan \theta = \frac{T}{W} \qquad W = \frac{T}{\tan \theta}$$

$$W_{max} = \frac{T}{\tan(\gamma + \phi)} \qquad \text{(A41)}$$

or

$$W_{max} = \frac{325.7 \text{ x HP x } f}{K \tan(\gamma + \phi)} \qquad \text{(A42)}$$

Converting the symbols used in equations (A41) and (A42) to those used generally, these equations become

$$F_{BZ} = \frac{T_{IB}}{\tan(i_B + \theta)} \qquad \text{(A43)}$$

or

$$F_{BZ} = \frac{325.7 \text{ (IHP x } f_s - \text{EHP)}}{V_{IB} \tan(i_B + \theta)} \qquad \text{(A44)}$$

Equations (A43) and (A44) indicate the maximum downward force possible under steady state icebreaking conditions. The term in parentheses in the numerator of equation (A44) indicates the horsepower available for breaking ice. It is to be recalled that f_s in this equation represents an overall efficiency of plant and propeller and varies between 10 and 25 percent.

In arriving at equations (A43) and (A44) it should be noted that Simonson made the following assumptions:

1. There are no momentum effects.
2. Friction with these was negligible*.
3. Thrust was directed horizontally at all times*.
4. The center of flotation remained a "pivot point".*
5. There is no change in displacement*.

Since friction was disregarded, the spread angle of the bow was not relevant and for that reason does not appear.

---

* These assumptions were not stated.

Furthermore it is interesting to note that Simonson felt "momentum should be neglected as it is desirable to break ice without charging or ramming". (12) His analysis was a basic approximation for the steady state condition.

Vinogradov's Equation

In a book published in 1946 a mathematical analysis of the downward force under the bow created during ramming was presented (13). The development was paraphrased and presented as an appendix to a paper presented by Ferris in 1959. (14)

The paraphrased version is presented here.

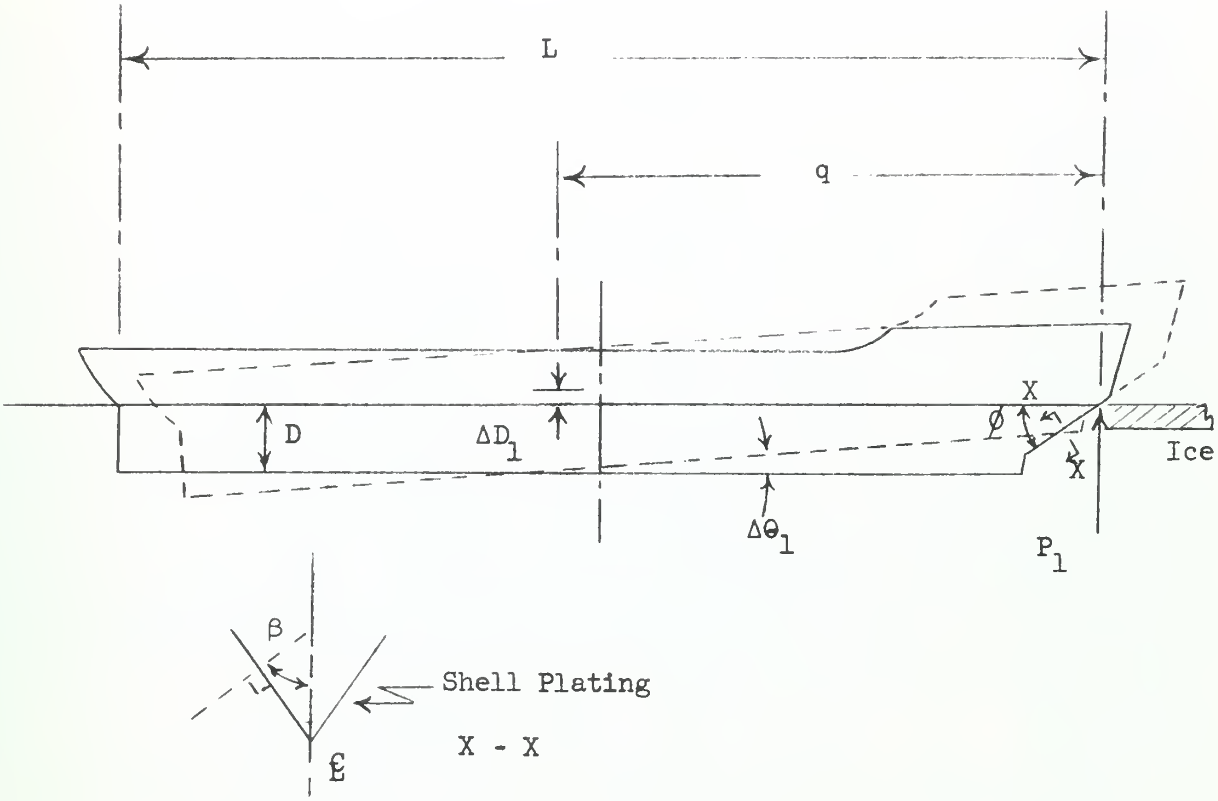
"The analysis is based on the following concept. An icebreaker moving with known velocity strikes a uniform ice shelf and the bow of the ship glides up until the downward pressure reaches a magnitude which causes the ice shelf to collapse. While the ship is climbing the ice shelf, the propellers continue to push. In general, the forward motion of the ship is not reduced to zero at the instant when the ice collapses.

The quantity which is to be determined is the maximum value of the vertical force P developed on the stem of the icebreaker. The maximum is reached at the instant when collapse of the ice shelf impends; therefore, the dynamic study will cover events occurring up to this time.

The principle of conservation of energy is applied. Energy expended is a portion of the ship's kinetic energy plus the propeller thrust acting through the distance travelled. The energy expended is diverted into three channels; (a) Energy dissipated by impact of the bow of the ship on the ice shelf; (b) potential energy of the ship due to its being raised and changed in trim; (c) frictional loss caused by running of ship against the ice shelf

Figure A-VII (14)

Illustration of Terms Used in Analysis by Vinogradov



L = Length between perpendiculars
B = Beam
W = Displacement
q = Distance from stem to center of flotation
D = Draft
ΔD_1 = Maximum change in draft
$\Delta\Theta_1$ = Maximum change in trim
ϕ = Angle of Stem to horizontal
α = Waterplane area coefficient
δ = Block coefficient
P_1 = Maximum value of the vertical reaction
S = Area of waterplane
γ = Density of sea water
m = Longitudinal metacentric height

$$(E_0 - E_1) + E_2 = E_3 + E_4 + E_5 \qquad \text{(A45)}$$

where

E_0 = kinetic energy of ship when the ice is first touched

E_1 = kinetic energy of ship when ice collapses

E_2 = energy derived from propeller thrust

E_3 = energy dissipated by impact

E_4 = potential energy acquired by ship

E_5 = energy lost by friction

"Let W represent the weight or displacement of the ship, v_0 the velocity when the ice is first touched, and v_1 the velocity at the instant when the ice collapses. The initial kinetic energy is then $E_0 = (W/2g)v_0^2$. The remaining kinetic energy at the instant of collapse is $E_1 = (W/2g)v_1^2$. Kinetic energy absorbed during the operation is

$$E_0 - E_1 = \frac{W}{2g}({v_0}^2 - {v_1}^2) \qquad \text{(A46)}$$

"The next item considered is the energy delivered by the propellers to the ship while the latter is sliding up on the ice shelf. During this interval there is a reduction in mean draft designated by ΔD_1, and the ship assumes an angle of trim of $\Delta\theta_1$. Distance from the point of contact on the stem to the center of flotation is designated by q. The stem of the ship is sloped at angle $\emptyset$ from horizontal. Then from the instant of first contact until the time when the ice collapses, the linear advance of the ship is

$$\Delta D_1 \cot \varphi + q\Delta\theta_1 \cot$$

Let T represent the average value of propeller thrust during this advance, then

$$E_2 = T(\Delta D_1 + q\Delta\Theta_1) \cot \varphi \qquad (A47)$$

"It is desired that this formula be expressed in different terms so as to include P_1, the maximum value of the vertical force developed at the stem. Assuming that P_1 is small in comparison with the displacement W, and that the change in draft and trim do not seriously change the properties of the waterplane, $\Delta D_1 = P_1/$ S, S being the waterplane area and the density of sea water. The angle of trim $\Delta\Theta_1$, depends on the applied moment P_1q, and the longitudinal metacentric height, m; thus $\Delta\Theta_1 = P_{1q}/Wm$. The energy under consideration can then be expressed as

$$E_2 = T \left[\frac{P_1}{S} + \frac{P_{1q}{}^2}{Wm}\right] \cos \varphi \qquad (A48)$$

"Waterplane area equals the product of length, beam and waterplane area coefficient, or $S = LB\alpha$. Weight of ship equals the product of length, beam, draft, block coefficient and density of sea water, or $W = LBD\delta$. New non-dimensional coefficients k_1 and k_2 are arbitrarily set up by relationships $q = k_1(L/2)$ and $m = (k_2{}^2\alpha^2L^2)/(D\delta)$, it being assumed that the longitudinal metacentric height is essentially equal to the height between center of buoyancy and longitudinal metacenter. Substituting these new quantities in the last equation, there results

$$E_2 = \left\{\frac{\delta}{\alpha}\left[1 + \left(\frac{k_1}{k_2}\right)^2 \frac{1}{4\alpha}\right]\right\} \frac{DP_1}{W} T \cot \varphi$$

or, for abbreviation

$$E_2 = A \frac{DP_1}{W} T \cot \varphi \qquad (A49)$$

"According to the theory of impact, when two bodies collide normally there is always a dissipation of energy, whose magnitude depends on relative velocity and a physical constant e known as the coefficient of restitution. Now the stem of the ship does not collide normally with the edge of the ice owing to the fact that the stem is sloped at an angle from the horizontal. The component of initial velocity v_0 which is directed normal to the edge of the ice is $v_0 \sin \varphi$ and the energy dissipated by impact is

$$E_3 = \frac{W}{2g} (v_0 \sin \varphi)^2 (1 - e^2) \qquad (A50)$$

"The vertical force P is a variable which keeps increasing as the ship slides up on the ice. The total rise of the point on the stem at which P is first applied equals to reduction in draft ΔD_1 plus the angle of trim, in radians, times the horizontal arm between center of flotation and stem $\Delta\Theta_1 q$. The potential energy set up by the force P is therefore

$$E_4 = \int_0^{\Delta D_1} P d\Delta D + \int_0^{\Delta\Theta_1} P q d\Delta\Theta \qquad (A51)$$

"Energy is dissipated by sliding friction between the shell plating and the ice. The coefficient of sliding friction is f and it must be applied to that component of the pressure which is normal to the plating. The resultant frictional force, designated by F, acts in a direction parallel to the stem of the ship and is a variable; half of it acts on one side of the stem and half on the other".

"The energy dissipated by frictional force F acting through a distance which is determined by the changes in draft and trim is given by

$$E = \frac{1}{\sin} \int_{0}^{\Delta D_1} F d\Delta D + \frac{1}{\sin} \int_{0}^{\Delta \theta_1} F q d\Delta\theta \qquad (A52)$$

"Consider an inclined plane intersecting the bow of the ship in a direction normal to the stem; this section of the bow will appear as a wedge with essentially flat sides and the normal pressure on these sides makes an angle β with the centerline plane."

"As the bow rides up on the ice shelf it forms a wedgelike groove, with pressure developed normal to the faces of the groove and friction along the faces of the groove directed parallel to the sloping stem".

"Let R be the resultant force acting normal to the stem. On each side, then, the force acting normal to the plating is

$$(R/2)(1/\cos\beta)$$

so the resultant frictional force is given by

$$F = fR\left(\frac{1}{\cos\beta}\right) \qquad (A53)$$

"The magnitude of force R is related to other forces acting on the ship as follows:

$$R = P\cos\varphi + T\sin\varphi \qquad (A54)$$

Equation (A53) is thus rewritten as

$$F = f\left[P\frac{\cos\varphi}{\cos\beta} + T\frac{\sin\varphi}{\cos\beta}\right] \qquad (A55)$$

"Substituting in equation (A52), there results

$$E_5 = f\,\frac{\cot \varphi}{\cos \beta} \int_0^{\Delta D_1} P d\Delta D$$

$$+ f\,\frac{T}{\cos \beta} \int_0^{\Delta D_1} d\Delta D + f\,\frac{\cot \varphi}{\cos \beta} \int_0^{\Delta \theta_1} Pq d\Delta \theta \qquad \text{(A56)}$$

$$+ \frac{fT}{\cos \beta} \int_0^{\Delta \theta_1} q d\Delta \theta$$

Equations (A51) and (A56) are combined, as follows:

$$E_4 + E_5 = \left(1 + f\,\frac{\cot \varphi}{\cos \beta}\right) \int_0^{\Delta D_1} P d\Delta D$$

$$+ f\,\frac{T}{\cos \beta} \int_0^{\Delta D_1} d\Delta D + \left(1 + f\,\frac{\cot \varphi}{\cos \beta}\right)$$

$$\int_0^{\Delta \theta_1} Pq d\Delta \theta + \frac{fT}{\cos \beta} \int_0^{\Delta \theta_1} q d\Delta \theta \qquad \text{(A57)}$$

For P the quantity SΔD can be substituted and for Pq the quantity WmΔθ."

"Rewriting (A57) gives

$$E_4 + E_5 = \left(1 + f\,\frac{\cot \varphi}{\cos \beta}\right) \quad S \int_0^{\Delta D_1} \Delta D d\Delta D$$

$$+ f \frac{T}{\cos \beta} \int_0^{\Delta D_1} d\Delta D + (1 + f \frac{\cot \varphi}{\cos \beta}) Wm$$

$$= \int_0^{\Delta \theta_1} \Delta\theta\Delta\theta + f \frac{Tq}{\cos \beta} \int_0^{\Delta\theta_1} d\Delta\theta \qquad (A58)$$

Integrating

$$E_4 + E_5 = (1 + f \frac{\cot \varphi}{\cos \beta}) \quad S (\frac{\Delta D_1^2}{2})$$

$$+ f \frac{T}{\cos \beta} (\Delta D_1) + (1 + f \frac{\cot \varphi}{\cos \beta}) Wm (\frac{\Delta\theta_1^2}{2}) + f \frac{Tq}{\cos \beta} (\Delta\theta_1) \qquad (A59)$$

"Since

$$\Delta D_1 = \frac{P_1}{S} \text{ and } \Delta\theta_1 = \frac{P_{1q}}{Wm}$$

$$E_4 + E_5 = (\frac{1}{S} + \frac{q^2}{Wm})$$

$$\left[(1 + f \frac{\cot \varphi}{\cos \beta}) \frac{P_1^2}{2} + f \frac{TP_1}{\cos \beta}\right] \qquad (A60)$$

"Using the previously established values, $S = LB\alpha$, $q = k_1(L/2)$, $W = LBD\delta$ and $m = (k_2^2\alpha^2L^2)/(D\delta)$,

$$E_4 + E_5 = \left\{\frac{\delta}{\alpha} \left[1 + (\frac{k_1}{k_2})^2 \frac{1}{4\alpha}\right]\right\} \frac{D}{W}$$

$$\left[(1 + f \frac{\cot \varphi}{\cos \beta}) \frac{P_1^2}{2} + f \frac{TP_1}{\cos \beta}\right]$$

or

$E_4 + E_5$

$$= \frac{AD}{W}\left[\left(1 + f\,\frac{\cot\varphi}{\cos\beta}\right)\frac{P_1^2}{2} + f\,\frac{TP_1}{\cos\beta}\right] \qquad (A61)$$

"Substituting all the values of component energies in equation (A45) there finally results

$$A\left[\left(\frac{P_1}{W}\right)^2 - 2X\,\frac{P_1}{W}\,\frac{T}{W}\right] = Y\,\frac{v_0^2\left[1 - (1 - e^2)\sin^2\varphi\right] - v_1^2}{gD} \qquad (A62)$$

in which

$$X = \frac{1 - \dfrac{f}{\cos\beta}\tan\varphi}{1 + \dfrac{f}{\cos\beta}\cot\varphi}\cot\varphi \qquad (A63)$$

$$Y = \frac{1}{1 + \dfrac{f}{\cos\beta}\cot\varphi} \qquad (A64)$$

The quantity to be calculated is the downward icebreaking force P_1. Solving the quadratic equation (A62) gives

$$P_1 = XT \pm \left\{X^2T^2 + \frac{Y}{A}W^2\,\frac{v_0^2\left[1 - (1 - e^2)\sin^2\varphi\right] - v_1^2}{gD}\right\}^{1/2} \qquad (A65)$$

The positive sign in front of the radical must be used in order for the value of P_1 to come out positive."

It is important to note the assumptions and expedients used by Vinogradov in his development.

1. Trim was taken into account for the solution of movement but it was disregarded in his solution for the resultant perpendicular to the stem R based on thrust T and the downward component P_1. See equation (A54).
2. Thrust T was directed horizontally at all times.
3. Thrust T was kept as a constant instead of attempting to elaborate and make it a function of other parameters such as i h p, propeller area, velocity, and so forth.
4. The friction loss is not correct in that the normal force on the bow plating would be valid only for static equilibrium.
5. There is no mention of the possibility that some of the kinetic energy while sliding up may be in the form of rotation.
6. It was assumed that the change in trim and draft did not seriously affect properties of the waterplane or the longitudinal metacentric height.
7. The change of trim is based on the original displacement using the equation for a couple when actually the displacement is effectively changed.
8. q is used exclusively as a constant representing the distance from the center of flotation to the forward perpendicular which is the original point of contact. For the determination of certain distances this is proper but it is an assumption when dealing with moment arms since the point of contact moves.

9. The expression for loss of energy on impact is based on direct central impact. In other words it is assumed that the loss is the same as if a perpendicular to the stem passed through the center* of gravity.
10. The normal assumption was utilized that GM (longitudinal) = BM (longitudinal).
11. The final equation is written including v_1 as the velocity during the sliding. However, in effect the equation is valid only when $v_1 = 0$ since there is not a continuous spectrum of velocity from v_0 to v_1.
12. A necessary step in his development was the use of static equilibrium, $F_R = 0$. See equation (A54). Acceleration at that point in contact with the ice in the direction of the force may be zero but not the acceleration of the body.

In summary, Vinogradov's equation states that the downward component of force, P_1, is a function of the following:

f, coefficient of sliding friction
φ, angle of stem, deg.
β, angle of normal to shell plating with respect to ℄ plane, deg.
δ block coefficient
α, waterplane coefficient
q, L/2 plus the distance aft from ⊗ to the center of flotation, ft.
L, length between perpendiculars, ft.
D, draft, ft.

m, longitudinal metacentric height, ft.

T, thrust, tons

W, displacement, tons

e, coefficient of restitution

v_o, speed just prior to impact, ft/sec

v_1, speed while sliding up, ft/sec (normally taken as zero to get maximum P_1).

Jansson's Equation

In 1956 Jansson presented an equation for the determination of work utilized in the ramming of ice (15). It does not indicate the downward force on the ice but is included here because of it's comprehensive approach.

Jansson used the following symbols for his development.

P = Vertical force between vessel and edge of ice.

T = Thrust of propeller, a function of speed.

X = Trim in deg. (change of trim).

W = Weight of vessel.

v = Speed of vessel.

M = Mass of vessel plus virtual mass of water.

J = Moment of inertia of mass of vessel plus virtual added mass of water, referred to a horizontal axis through the center. of gravity and at right angles to the lateral plane.

S = Length coordinate in meters.

Y = Vertical coordinate in meters.

ω = Angular velocity about a horizontal axis at right angles to the lateral plane.

p = Number of tons load for 1 meter immersion.

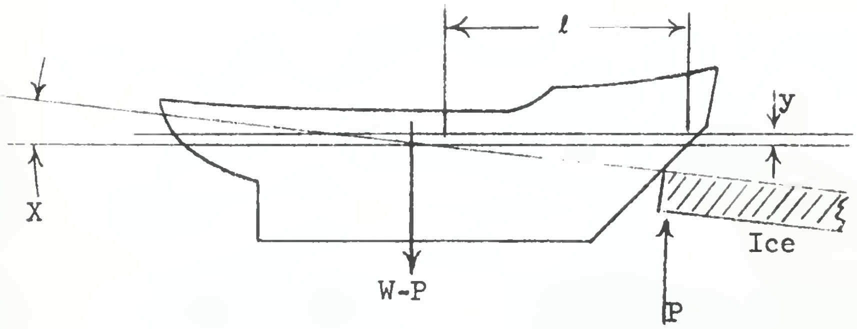
q = Trim moment in ton-meters for 1 radian trim.

ℓ = Distance from center of gravity of waterline areas to foremost point in the water line in meters.

As can be seen from Figure A-VIII , the following equation can be determined statically:

Figure A-VIII (15)

An Icebreaker with Stem in Contact with Edge of Ice



$$P = p\,y \quad . \tag{A66}$$

where y is the change in mean draft in meters.

$$P \cdot \ell = q \cdot x \tag{A67}$$

where x is in radians.

It is assumed that trim is small enough to assume p, ℓ, and q may be taken as constant.

Equations (A66) and (A67) may be combined to get

$$y = \frac{qx}{p\ell} \tag{A68}$$

Writing down the energy equation for condition 1 (immediately before raumming) and condition 2 (as the bow slides up the ice) the following equation is obtained.

$$\frac{1}{2} M \left(v_1^2 - v_2^2\right) + \int_{S_1}^{S_2} T\,ds =$$

$$\int_{y_1}^{y_2} p\,y\,dy + \int_{x_1}^{x_2} q\,x\,dx + \frac{1}{2} J \left(\omega_2^2 - \omega_1^2\right) \tag{A69}$$

In his development he deliberately neglected the friction between the ice and the forward end of the vessel. Furthermore, without mentioning it, he has assumed that no energy is lost on impact.

When the maximum vertical force is reached the angle of trim, X, has reached its maximum value and the speed, v, is zero. Thus the angular velocity, ω, is zero. For initial conditions he uses $y_{,} = 0$, $X_1 = 0$, and

and $\omega_1 = 0$. Equation (A69) reduces to

$$\frac{1}{2} M v_1^2 + \int_{S_1}^{S_2} T \, ds = \frac{p y_2^2}{2} + \frac{q x_2^2}{2} \tag{A70}$$

Equation (A68) can be substituted into equation (A70) to obtain the following expression for maximum icebreaking work:

$$\frac{1}{2} M v_1^2 + \int_{S_1}^{S_2} T \, ds = X_2^2 \left(\frac{q^2}{2 \, p \, \ell^2} \right) \tag{A71}$$

It is noted that the term in parentheses is constant for a given ship.

Although Jansson does not go further it would be possible to solve this equation for trim, X, if T(s) were known as well as S_1 and S_2. Substitution of X_2 back into equation (A67) would then yield the maximum downward force.

It is important to reiterate that the result would have neglected friction and impact losses.

Richardson's Equation

In 1959 Richardson presented an equation for the downward force under the bow created during ramming. (75) The development is the most complete to this date (1964) and is part of a model study of the force system.

The complete development will not be reproduced here. The steps are basically the same as those of Vinogradov. It is based on the conservation of energy and is shown as follows:

$$T_V - T_f + T_T = T_S + T_{P,\psi} + T_R + T_\nu \qquad \text{(A72)}$$

where

T_V = the kinetic energy at the instant of initial contact with the ice shelf.

T_f = the kinetic energy remaining after the ice splits.

T_T = the energy furnished by the propellers from the instant of contact up to the moment the ice splits or breaks, or as the case may be, the forward motion closes.

T_S = the energy lost at the impact of stem with the ice shelf.

$T_{P,\psi}$ = the energy spent to raise and trim the icebreaker.

T_R = the energy spent in friction between the hull and the ice.

T_ν = the energy spent in overcoming the friction and wave resistance from the instant of contact up to the moment the ice splits or breaks M motion ceases.

T_E = the energy lost in elastic vibrations. (This loss is neglected.)

The use of this approach is quite appropriate. It agrees with Vinogradov except that Richardson has wisely included a term for non-ice friction and wave making resistance. He also takes virtual mass into account.

The equation for downward force is as follows:

$$P_V = P_h \pm \sqrt{P_h^2 \quad {}^2 + \frac{\Delta}{Ag\gamma}\left\{v_1^2 \left[(1+\psi_x)-(1-\epsilon^2)\sin^2\varphi\right]-v_2^2(1+ \quad _x)\right\}} \qquad \text{(A73)}$$

where

$$= \frac{\dfrac{\tau}{P} - \dfrac{c \tan\alpha}{\cos\beta} - \dfrac{\overline{R_{rt}}}{P}}{1 + \dfrac{c \cot\alpha}{\cos\beta}} \cot\alpha \qquad \text{(A74)}$$

$$\eta = 1 + \frac{c \cot\alpha}{\cos\beta} \qquad \text{(A75)}$$

$\overline{R_{rt}}$ = arithmetic mean for ship resistance computed for v_1, and v_2.

τ = propeller thrust

$$A = \left(\frac{1}{p} + \frac{i^2}{q^2}\right) \qquad \text{(A76)}$$

ψ_x = added virtual mass (percent)

P_h = horizontal component of force produced by the icebreaker.

C = coefficient of friction between the hull and the ice.

D = displacement

L = length at load waterline

r = distance from center of rotation of the waterplane from the point of contact on the ice.

φ = angle of the stem measured from the load waterline.

ψ = angle of trim

α = $\varphi + \psi$. (Since ψ is of the order of 1 to 2 degrees, $\alpha = \varphi$ will be used in most trigonometric quantities, i.e. $\sin\alpha \approx \sin\varphi$

v_1 = velocity of the icebreaker at instant of impact with the ice.

v_2 = velocity at the close of the cycle, i.e. at the instant the ice breaks or zero if the icebreaker comes to a dead stop before the ice breaks.

p = tons per inch immersion at the load waterline.

q = moment in foot tons per inch trim.

The following assumptions or expedients were used by Richardson:

1. In effect, all steps lead to the final condition of $v_2 = 0$. The equation is not valid where acceleration may exist.
2. It was assumed that the change in trim and draft did not seriously effect properties of the waterplane or moment to change trim one inch.
3. The distance from the center of rotation to the point of ice contact is assumed constant.
4. The "center of rotation" as he uses it is the center of flotation. In absolute terms this is not actually the case since there is also a change in draft (and effective displacement).
5. Although equations include trim angle in the first part of the development it is effectively dropped when he equates angle of trim plus angle of stem to angle of stem. It is granted that when the cosine is used there would be little difference but this is not necessarily the case for sine and tangent.

6. In the end conditions he uses rotational velocity (ω) equal to zero but this would only be true when all kinetic energy is lost or converted.
7. Although he recognizes that thrust may not always be horizontal only the work developed horizontally by this force is incorporated.
8. The change in the vertical position of the center of gravity is assumed to be the same as the average change in draft.
9. The angle of trim throughout the transition is based on static equilibrium.
10. The expression for loss of energy on impact is based on direct central impact. In other words it is assumed that the loss is the same as if a perpendicular to the stem passed through the center of gravity.
11. The determination of the downward force throughout the transition is based on static equilibrium.
12. The horizontal component of force against the ice must be known to use the equation. As used this is not the same as thrust and it is not clear where this value comes from.

General Dynamics Equation

As part of a report released in 1959 concerning the feasibility of a nuclear icebreaker, an equation was developed representing the relative magnitude of the force "transmitted to the ice" at the bow. (17) However, this equation does not give the direction of this force and only a component of it is downward.

The following symbols were used in the development:

T = Thrust at zero speed in lb.

ρ = Mass density, lb $\sec^2/\text{ft}^4$

α = Angle at the bow in the vertical plane, deg.

2B = Angle at the bow in the horizontal plane in deg. (Note that this is not the same as 2β).

Δ = Vessel displacement in tons.

P = Shaft horsepower.

A = Propeller disc area, ft^2.

γ = Change of trim in deg.

W = Weight supported by ice.

R = Force perpendicular to the stem.

The forces acting include thrust at zero speed

$$T = \left[\left(\frac{\rho}{g}\right) 2 P^2 A \right]^{1/3} \qquad \text{(A77)}$$

(In this form the units are not compatible and this is not explained.) and that portion of the weight supported by the ice

$$W = 2 \Delta \tan \gamma \qquad \text{(A78)}$$

(This equation is from Simonson (12).

Figure A-IX (17)

Geometry at an Icebreaker Bow

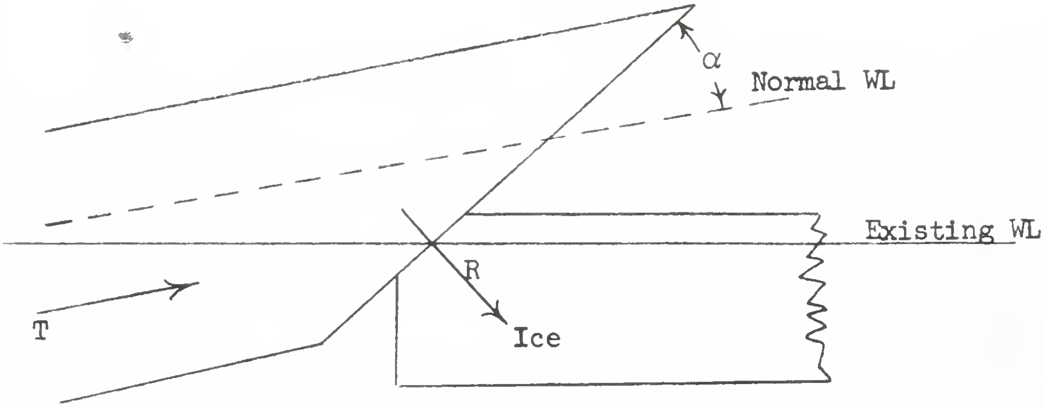
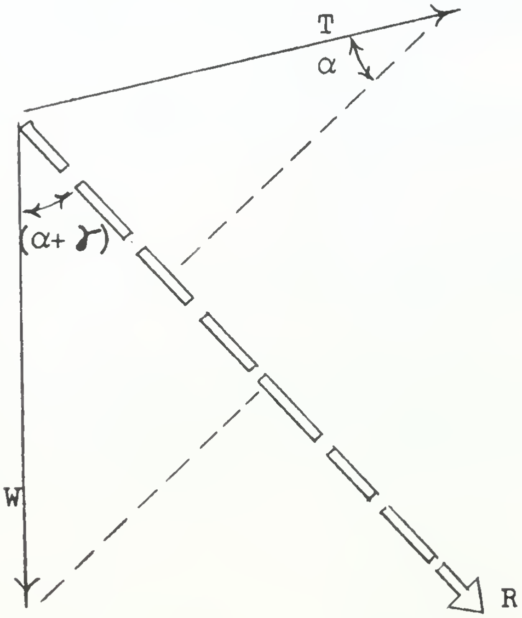


Figure A-X

Forces Acting on Ice from Bow



In substance Figures A-IX and A-X appear in the reference (17). However, they have been illustrated here with a fair amount of clarification and simplification.

It is interesting to note that this is the only development to date (1964) which assumes that thrust remains parallel to the normal waterline (base line) at all times.

Neglecting friction the force, R, perpendicular to the stem can be deduced from Figure A-X.

$$R = T \sin \alpha + W \cos (\alpha + \gamma) \qquad \text{(A79)}$$

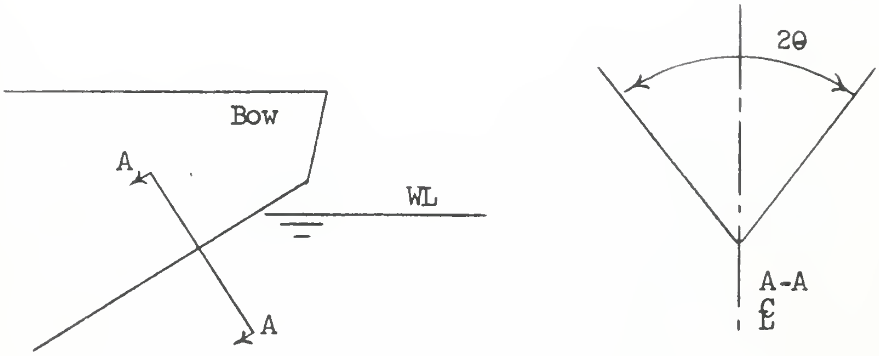
Using the analogy of a wedge being forced into the ice by force, R, (neglecting friction), an equation can be developed using the term (2θ) as the "wedge angle". Figure 21B in the reference is a three dimensional representation which is quite confusing and for that reason is not shown here. However, the definition of the "wedge angle" is needed. Although it is not explained, it is apparent from its use that it is the spread angle seen as one looks down the stem. See Figure A-XI.

The force transmitted to the ice is perpendicular to each side of the bow and is called R_1, where

$$R_1 = \left[\left({}^{\rho}/_{g}\ 2P^2 A \right)^{1/3} \sin \alpha + 2\Delta \tan \gamma \cos (\alpha + \gamma) \right] \left[\frac{\sqrt{\sin^2 \alpha + \tan^2 B}}{2 \tan B} \right] \qquad \text{(A80)}$$

The reference carries a graph of this force, R_1, versus bow angle, α. Figure A-XII is not a reproduction of this graph but does illustrate its appearance.

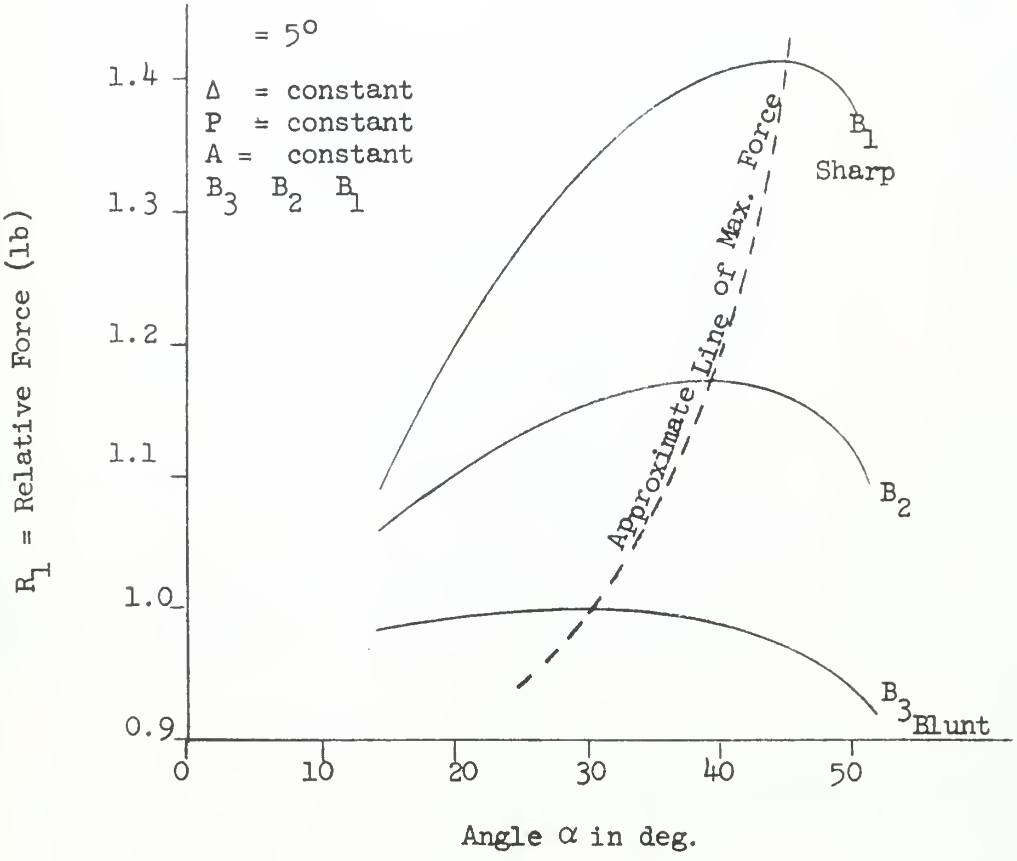
Figure A-XI
Illustration of "Wedge Angle"



"Wedge Angle" $\equiv 2\,\theta$

Figure A-XII

Illustration of a Plot entitled
"Variation of Force Exerted on
the Ice as a Function of Bow Design"



A reader must be extremely careful not to jump to any conclusions. The graph is simply a plot of the results of equation (A80) where trim angle α, displacement Δ, power P, and disc are A are held constant. It simply illustrates that if the bow is made sharper and all other parameters are held constant the force normal to the hull plating will increase. As a matter of interest and fact, the downward force, W, is constant throughout under the conditions used since Δ and γ are held constant!

Milano's Equation

In 1962 an equation was published giving the vertical bow reaction force which an icebreaker can develop climbing onto the ice (18).

The following symbols appear in the equation:

F = Downward vertical force.

μ_o = Coefficient of friction (dynamic) between steel and ice.

α = Angular rise of the forefoot.

β = Angle between the centerline plane and the normal to the shell at the bow.

X and Z are each a direct function of $_o$, α, and β.

T_o = Bollard pull, thrust

W = Vessel displacement

v = Velocity prior to contact with ice.

H = Draft

As presented the equation is as follows:

$$F = 0.91\,X\,T_o + \sqrt{0.828\,X^2\,T_o^2 + \frac{W^2 v^2 Z}{4g\,H}} \qquad \text{(A81)}$$

This equation originates from Vinogradov (13) but it has been abbreviated by selection of constants and coefficients. For example

$$T_{\text{VINOGRADOV}} = 0.91\,T_o$$

The equation is based on the same assumptions Vinogradov made and has the same limitations.

B. DETAILS OF PROCEDURE

General

When an icebreaker encounters virtually unyielding ice (rams) it crushes the ice locally to accomodate the bow, the bow then slides up on the ice with decreasing velocity, and then the icebreaker undergoes minor settling after the velocity of the bow relative to the ice has come to zero. At this last point the ship tends to slide back but is frequently held by static friction and/or forward thrust.

The following definitions will be of use for the purpose of constructing a mathematical model:

State 1. Immediately prior to contact with the ice.

$$t = 0$$

$$\dot{x} = 0 \qquad \dot{z} = 0 \qquad \dot{\theta} = 0$$

$$\ddot{x} = v_1 \qquad \ddot{z} = 0 \qquad \ddot{\theta} = 0$$

$$x = 0 \qquad z = 0 \qquad \theta = 0$$

$$F_{BZ} = 0 \qquad F_{BZ} = 0$$

Crushing Phase. Ice is being crushed locally to accomodate the bow. (The ice is not collapsing.)

During the crushing phase five equations may be expressed.

Vertical force at bow. (function of penetration)

Horizontal force at bow. (function of penetration)

Summation of Horizontal Forces

Summation of Vertical Forces

Summation of Moments

There are five time-dependent unknowns,

x, z, θ, Vertical force at bow (F_{BZC}), and Horizontal force at bow (F_{BXC}).

State 2. Local crushing has ceased and sliding without crushing commences. This is reached when the velocity of a point on the bow has a direction which is the same as the slope of the bow plus the trim. In other words, there is no component of bow velocity perpendicular to the stem.

$t = t_2$ (for crushing) $t = 0$ (for sliding)

$x = x_2 \qquad z = z_2 \qquad \theta = \theta_2$

$\dot{x} = \dot{x}_2 \qquad \dot{z} = \dot{z}_2 \qquad \dot{\theta} = \dot{\theta}_2$

$\ddot{x} = \ddot{x}_2 \qquad \ddot{z} = \ddot{z}_2 \qquad \ddot{\theta} = \ddot{\theta}_2$

$F_{BZ} = F_{BZ2} \qquad F_{BX} = F_{BX2}$

Sliding Phase. The bow slides up on the ice without further penetration.

During the sliding phase four equations may be expressed.

Equation of geometry since point of contact is fixed relative to the ice.

Summation of Horizontal Forces.

Summation of Vertical Forces.

Summation of Moments

There are four time-dependent unknowns, x, z, θ, and the force at the bow (which can be divided into two components).

State 3. The velocity of the bow relative to the ice has come to zero. (Velocities ($\dot{x}$, $\dot{z}$, $\dot{\theta}$) may be negligible but they are not necessarily zero.)

$$t = t_3$$

$$x = x_3 \qquad z = z_3 \qquad \theta = \theta_3$$

$$\dot{x} = \dot{x}_3 \qquad \dot{z} = \dot{z}_3 \qquad \dot{\theta} = \dot{\theta}_3$$

$$\ddot{x} = \ddot{x}_3 \qquad \ddot{z} = \ddot{z}_3 \qquad \ddot{\theta} = \ddot{\theta}_3$$

$$F_{BZ} = F_{BZ3} \qquad F_{BX} = F_{BX3}$$

State 4. The icebreaker is in static equilibrium. All velocities have become zero.

$$x = x_4 \qquad (x_4 = x_3) \; z = z_4 \qquad \theta = \theta_4$$

$$\dot{x} = 0 \qquad \dot{z} = 0 \qquad \dot{\theta} = 0$$

$$\ddot{x} = 0 \qquad \ddot{z} = 0 \qquad \ddot{\theta} = 0$$

$$F_{BZ} = F_{BZ4}$$

F_{BZ4} is the relatively sustained downward force under the bow we are seeking.

Bow Forces During Crushing

Assume all forces from the ice act on the bow at point A, the point of contact at the waterline.

If $\frac{N}{2}$ represents the force normal to the plating on each side, then the friction force can be represented by

$$\frac{F}{2} = f_k \frac{N}{2} \qquad \text{where } f_k = \text{coeff. of kinetic friction} \qquad \text{(B1)}$$

Note that the friction forces during crushing are parallel to the stem and perpendicular to the stem (each in the plane of the plating). This is because there is a component of velocity relative to the ice in each direction (parallel and perpendicular). See Figures B-I and B-II.

From Figure B-II it can be seen that

$$P = N \cos \beta + F \sin \beta \qquad \text{(B2)}$$

where β = angle between normal to plating and centerline plane.

Substitution of (B1) into (B2) leads to

$$P = N (\cos \beta + f_k \sin \beta) \qquad \text{(B3)}$$

From (B1) and (B3) we get

$$N = \frac{F}{f_k}$$

$$N = \frac{P}{\cos \beta + f_k \sin \beta} \qquad \text{(B4)}$$

$$\frac{F}{f_k} = \frac{P}{\cos \beta + f_k \sin \beta}$$

Figure B-I

Forces Acting on Bow During Crushing

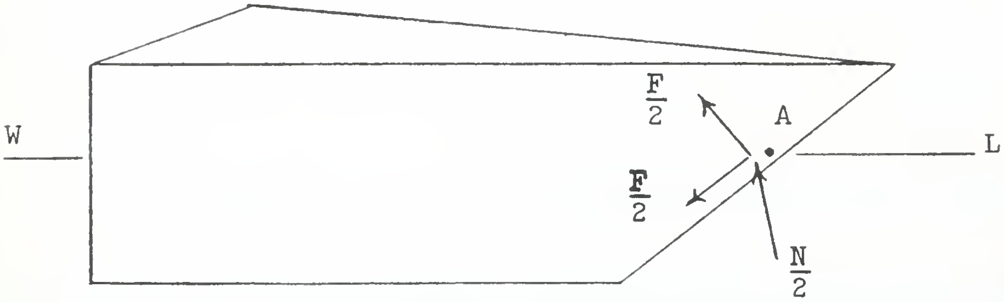
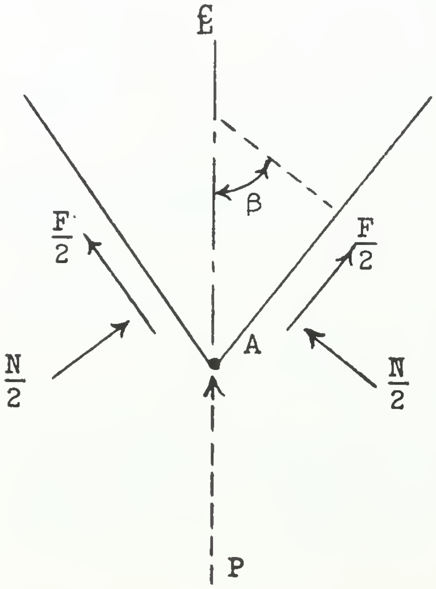


Figure B-II

Resolution of Friction and Normal Forces During Crushing(Looking Down Stem)



Set $k_1 = \dfrac{f_k}{\cos\beta + f_k \sin\beta}$ (B5)

Then $F = k_1 P.$ (B6)

The upward forces at the bow, designated F_{BZC}, can be seen in Figures B-III and B-IV.

$$F_{BZC} = P\cos(i_B + \theta) - F\sin(i_B + \theta) \qquad (B7)$$

$$F_{BZC} = N(\cos\beta + f_k\sin\beta)\cos(i_B + \theta) - Nf_k\sin(i_B + \theta)$$

The horizontal force to the left, designated F_{BXC}, can also be seen.

$$F_{BXC} = P\sin(i_B + \theta) + F\cos(i_B + \theta)$$

$$F_{BXC} = N(\cos\beta + f_k\sin\beta)\sin(i_B + \theta) + Nf_k\cos(i_B + \theta) \qquad (B8)$$

While crushing is taking place, assume that the ice is failing in compression over an area in contact with the bow plating. If the area in contact on each side of the bow is A/2 and the failing compressive stress of the ice is designated σ, then

$$\frac{N}{2} = \sigma\frac{A}{2} \qquad (B9)$$

As can be seen in Figures B-VI and B-VII,

$$\frac{A}{2} = \frac{A'}{2\sin\beta}$$

$$\frac{A'}{2} = \frac{1}{2}\left[x - (\text{corr. for } z \text{ and } \theta)\right]^2 \tan(i_B + \theta) \qquad (B10)$$

Assume that area-triangle remains at point A at bow (intersection of waterline and stem) and is small enough (or that ice is deep enough) to keep

Figure B-III

Bow Forces During Crushing

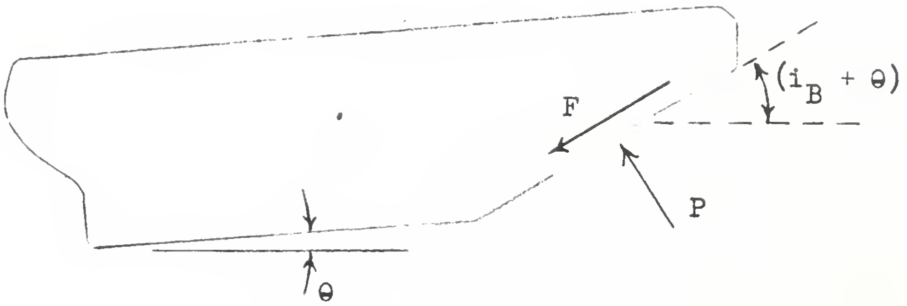


Figure B-IV

Bow Forces During Crushing Resolved
into X and Z Directions

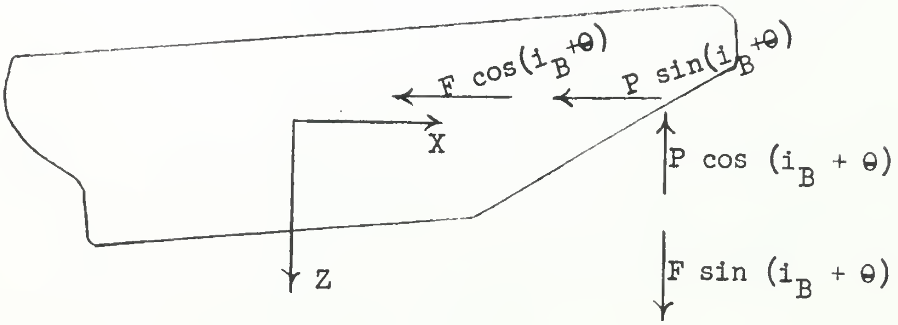


Figure B-V

Coordinate System Defined by Position
when t = 0, Immediately Prior to Contact

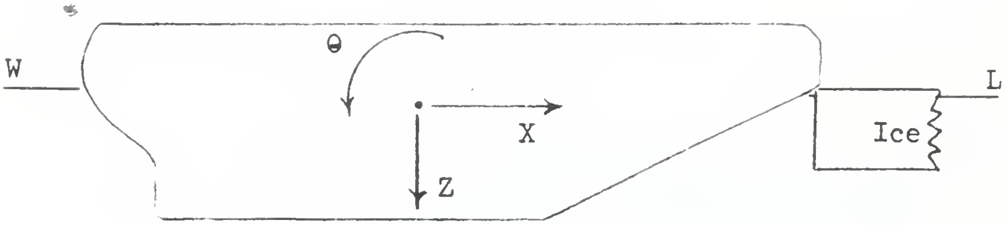
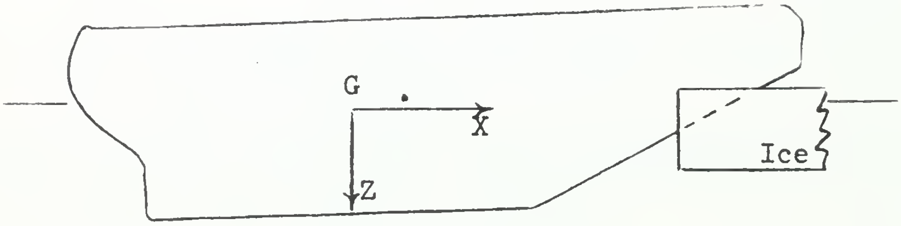


Figure B-VI

Local Crushing of Ice



shape triangular.

In order to correct for trim, θ, it is necessary to define distance from G to A. Assume, for this purpose, that KG (height of center of gravity above keel) is about the same as H (draft).

$$(GA)\ \text{Horizontal} = \left(\frac{L}{2} - LCG\right)$$

where L = Length between perpendiculars.

LCG = Distance from midships to center of gravity, + if forward, - if aft.

In Figure B-VIII it can be seen that

$$(\text{Corr. for } z \text{ and } \theta) = \frac{\left(\frac{L}{2} - LCG\right)\theta - z}{\tan\,(i_B + \theta)} \qquad \text{(B11)}$$

From Figures B-VII and B-VIII, and from equations (B10 and (B11), it can be seen that

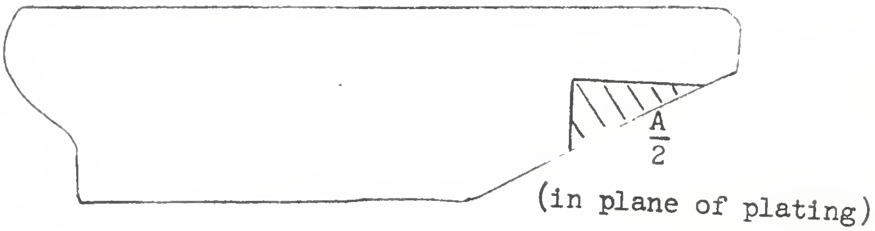
$$\frac{A'}{2} = \frac{1}{2}\left[x - \left(\frac{\left(\frac{L}{2} - LCG\right)\theta - (z)}{\tan\,(i_B + \theta)}\right)\right]^2 \tan\,(i_B + \theta)$$

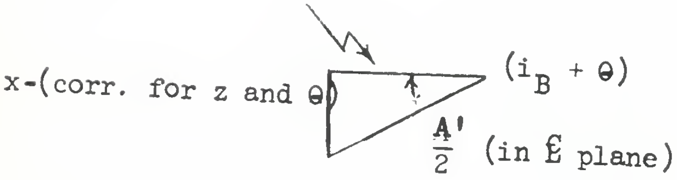
$$\frac{A'}{2} = \frac{1}{2}\left[x \tan\,(i_B + \theta) - \left(\frac{L}{2} - LCG\right)\theta + z\right]^2 \frac{1}{\tan\,(i_B + \theta)} \qquad \text{(B12)}$$

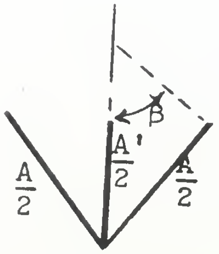
$$N = \sigma A = \frac{\sigma}{\sin\beta \tan\,(i_B + \theta)}\left[x \tan\,(i_B + \theta) - \left(\frac{L}{2} - LCG\right)\theta + z\right]^2$$

Substitution of equation (B12) into equation (B3) leads to the force in the upward direction at the bow during crushing.

Figure B-VII
Area in Contact During Crushing



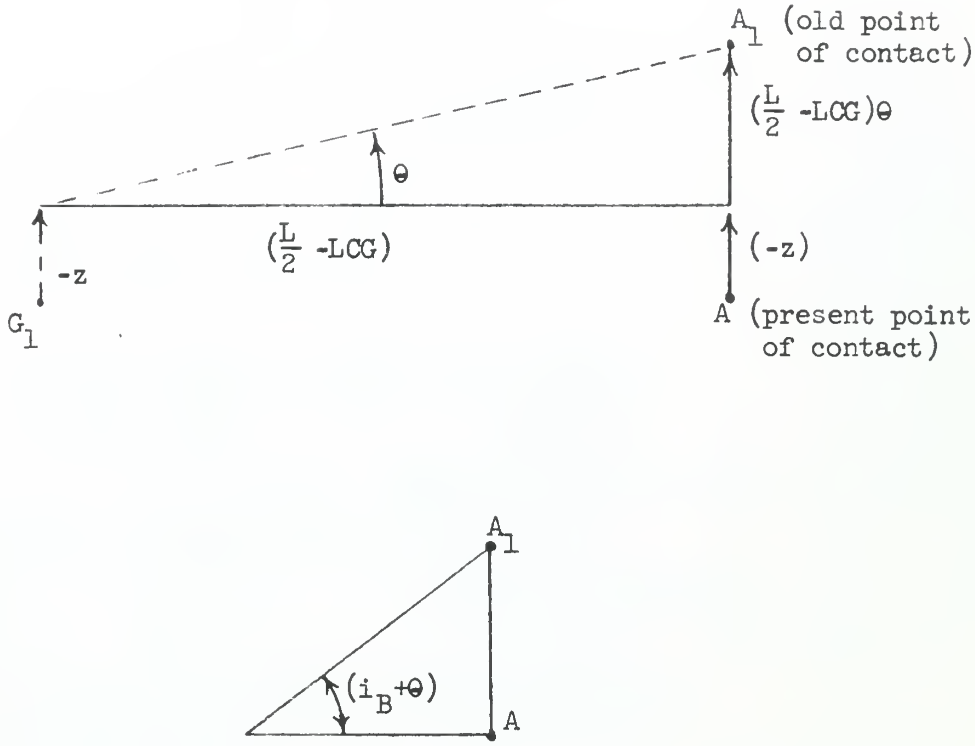




Looking Down Stem

Figure B-VIII

Movement of A_1 for Purpose of Area Correction


A_1 (old point of contact)
$(\frac{L}{2} - LCG)\theta$
θ
$(\frac{L}{2} - LCG)$
$(-z)$
$-z$
G_1
A (present point of contact)
A_1
$(i_B+\theta)$
A

(Correction for z and θ)

$$F_{BZC} = \frac{\sigma (\cos\beta + f_k \sin\beta) \cos (i_B + \theta)}{\sin \beta \tan (i_B + \theta)} \left[x \tan (i_B + \theta) - (\frac{L}{2} - LCG)\theta + z \right]^2$$

$$\frac{- \sigma f_k \sin (i_B + \theta)}{\sin \beta \tan (i_B + \theta)} \left[x \tan (i_B + \theta) - (\frac{L}{2} - LCG)\theta + z \right]^2$$

$$(a - b + c)(a - b + c) = a^2 - ab + ac - ab + b^2 - bc$$

$$+ac - bc + c^2$$

$$= a^2 - 2ab + b^2 - 2bc + 2ac + c^2$$

$$a = x \tan (i_B + \theta)$$

$$b = (\frac{L}{2} - LCG)\, \theta$$

$$c = z$$

$$\left[x \tan (i_B + \theta) - (\frac{L}{2} - LCG)\, \theta + z \right]^2 =$$

$$\left[x^2 \tan^2 (i_B + \theta) - 2 (\frac{L}{2} - LCG)\theta\, x \tan (i_B + \theta) \right.$$

$$\left. x (\frac{L}{2} - LCG)^2 \theta^2 - 2 z (\frac{L}{2} - LCG)\theta + 2 z x \tan (i_B + \theta) + z^2 \right]$$

$$F_{BZC} = \frac{\sigma (\cos \beta + f_k \sin \beta) \cos (i_B + \theta)\, x^2 \tan^2 (i_B + \theta)}{\sin \beta \tan (i_B + \theta)}$$

$$\frac{-2\sigma (\cos \beta + f_k \sin \beta) \cos (i_B + \theta) (\frac{L}{2} - LCG)\, \theta\, x \tan (i_B + \theta)}{\sin \beta \tan (i_B + \theta)}$$

$$+ \frac{\sigma (\cos \beta + f_k \sin \beta) \cos (i_B + \theta) (\frac{L}{2} - LCG)^2\, \theta^2}{\sin \beta \tan (i_B + \theta)}$$

$$\frac{-2 \sigma (\cos \beta + f_k \sin \beta) (i_B + \theta)\, z\, (\frac{L}{2} - LCG)\theta}{\sin \beta \tan (i_B + \theta)}$$

$$+\frac{2\sigma(\cos\beta + f_k \sin\beta)\cos(i_B+\theta) z\, x \tan(i_B+\theta)}{\sin\beta \tan(i_B+\theta)}$$

$$+\frac{\sigma(\cos\beta + f_k \sin\beta)\cos(i_B+\theta)\, z^2}{\sin\beta \tan(i_B+\theta)}$$

$$-\frac{\sigma f_k \sin(i_B+\theta)\, x^2 \tan^2(i_B+\theta)}{\sin\beta \tan(i_B+\theta)}$$

$$+\frac{2\sigma f_k \sin(i_B+\theta)\left(\frac{L}{2} - LCG\right)\theta\, x \tan(i_B+\theta)}{\sin\beta \tan(i_B+\theta)}$$

$$-\frac{\sigma f_k \sin(i_B+\theta)\left(\frac{L}{2} - LCG\right)^2 \theta^2}{\sin\beta \tan(i_B+\theta)}$$

$$+\frac{2\sigma f_k \sin(i_B+\theta)\, z\left(\frac{L}{2} - LCG\right)\theta}{\sin\beta \tan(i_B+\theta)}$$

$$-\frac{2\sigma f_k \sin(i_B+\theta)\, z\, x \tan(i_B+\theta)}{\sin\beta \tan(i_B+\theta)}$$

$$-\frac{\sigma f_k \sin(i_B+\theta)\, z^2}{\sin\beta \tan(i_B+\theta)} \qquad \text{(E13)}$$

It is necessary to linearize equation (E13) as much as possible in order to make it useful for inclusion in simultaneous differential equations.

Throughout this development, since θ is small (around 5° or less),

$\cos\theta = 1.00$

$\tan\theta = \theta$ rad. $\qquad$ $\sin\theta = \theta$ rad.

Function	Substitution	Limit of θ° if Error $<$ 1 o/o
$\cos\theta$	1.00	5.7°
$\tan\theta$	θ rad.	9.9°
$\sin\theta$	θ rad.	14.0°

Using fundamental trigonometric identities, the following conversions may be used.

$$\sin(i_B + \theta) = \sin i_B \cos\theta + \cos i_B \sin\theta$$

$$\sin(i_B + \theta) = \sin i_B + \theta \cos i_B$$

$$\cos(i_B + \theta) = \cos i_B \cos\theta - \sin i_B \sin\theta$$

$$\cos(i_B + \theta) = \cos i_B - \theta \sin i_B$$

$$\tan(i_B + \theta) = \frac{\sin(i_B + \theta)}{\cos(i_B + \theta)} = \frac{\sin i_B + \theta\cos i_B}{\cos i_B - \theta \sin i_B}$$

In order to check the orders of magnitude of equation (B13), let us examine the following terms:

$$(\frac{L}{2} - LCG)^{x^2}\theta x$$

$$(\frac{L}{2} - LCG)^2\theta^2$$

$$(\frac{L}{2} - LCG)\theta z$$

$$x z$$

$$z^2$$

It is noted that there is an initial velocity in the x-direction.

$$(\frac{dx}{dt}) = v_1$$

However,

$$(\frac{d\theta}{dt}) = 0 \quad \text{and} \quad (\frac{dz}{dt}) = 0$$

Up to the point where the velocity at the bow has a direction defined by $(i_B + \theta)$, the following magnitudes would be typically representative:

$$x \cong 10 \text{ ft}$$

$$\left(\frac{L}{2} - LCG\right)\theta + (-z) \cong 0.8 \text{ ft}$$

Assume $\left(\frac{L}{2} - LCG\right)\theta \cong 0.4 \text{ ft}$

$$-z \cong 0.4 \text{ ft}$$

Then, $x^2 = 100 \text{ ft.}$

$$\left(\frac{L}{2} - LCG\right)\theta\, x = 4 \text{ ft.}$$

$$\left(\frac{L}{2} - LCG\right)^2\theta^2 = 0.16 \text{ ft.}$$

$$\left(\frac{L}{2} - LCG\right)\theta\, z = 0.16 \text{ ft.}$$

$$x\, z = 4 \text{ ft.}$$

$$z^2 = 0.16 \text{ ft.}$$

When it becomes desirable to simplify equation (B13) it is apparent that all terms may be dropped except those containing x^2. It may furthermore be assumed that

$$\tan\,(i_B + \theta) \cong \tan\, i_B$$

during the crushing phase.

The simplifications mentioned above may be used directly to rewrite equation (B12) as follows:

$$N = \frac{\sigma\; x^2 \tan\, i_B}{\sin\, \beta}$$

Equation (B13) may now be written in the following form:

$$F_{BZC} = \left[\frac{\sigma \tan i_B}{\sin \beta} \cos i_B (\cos \beta + f_k \sin \beta) - \frac{\sigma \tan i_B}{\sin \beta} f_k \sin i_B \right] x^2 \qquad \text{(B14)}$$

Equation (B8) for the horizontal force at the bow may now be written in the following form:

$$F_{BXC} = \left[\frac{\sigma \tan i_B}{\sin \beta} \sin i_B (\cos \beta + f_k \sin \beta) + \frac{\sigma \tan i_B}{\sin \beta} f_k \cos i_B \right] x^2 \qquad \text{(B15)}$$

Free Body Description During Crushing

Figure B-IX shows the complete free body diagram for an icebreaker during the crushing phase. (It is the same for the sliding phase except for the composition of the bow forces.)

Point A (during crushing) is at the intersection of the waterline and is therefore fixed only in the z-direction; it is not fixed in the x-direction.

Since the origin of the coordinate system is at the position G had just prior to initial contact (See Figure B-V), the vertical moment arm from G to A is

$$(G\ A\ )_z = H - K\ G + z \qquad \text{(B16)}$$

where H is the initial draft and K G is the height of the center of gravity above the keel.

The horizontal moment arm from G to A, $(G\ A)_x$ is somewhat more complex since A is not absolutely fixed in the x-direction.

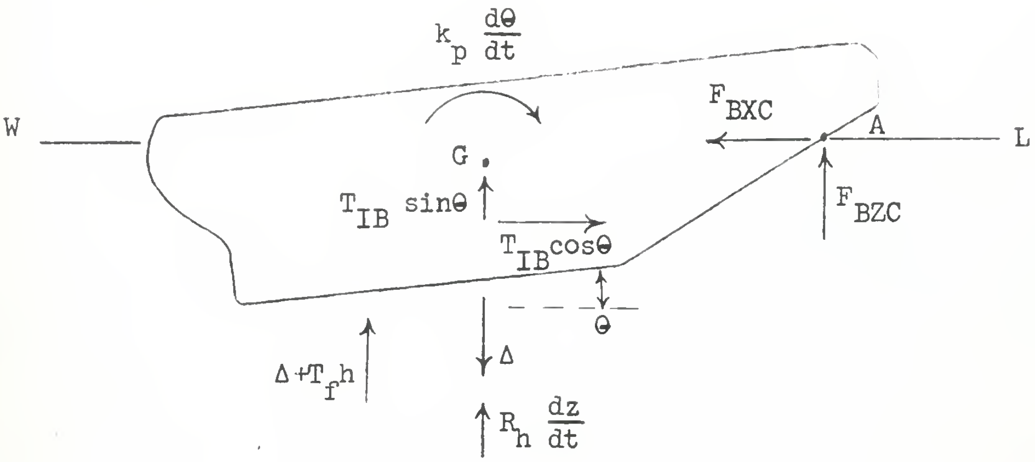
As can be seen in Figures B-XI and B-XII, the horizontal moment arm can be expressed as follows:

$$\begin{aligned}(G\ A)_x = &\left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right] \\ &- \Big[(H - KG) + z\Big]\,\theta \\ &- \frac{1}{\tan\,(i_B + \theta)}\left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right]\theta \qquad \text{(B18)}\end{aligned}$$

Linearize $(GA)_x$. First, linearize $\dfrac{1}{\tan\,(i_B + \theta)}$.

Figure B-IX

Free Body Diagram During Crushing Phase



$h \equiv$ increase of draft at LCF

$k_p \equiv$ coefficient of pitch damping

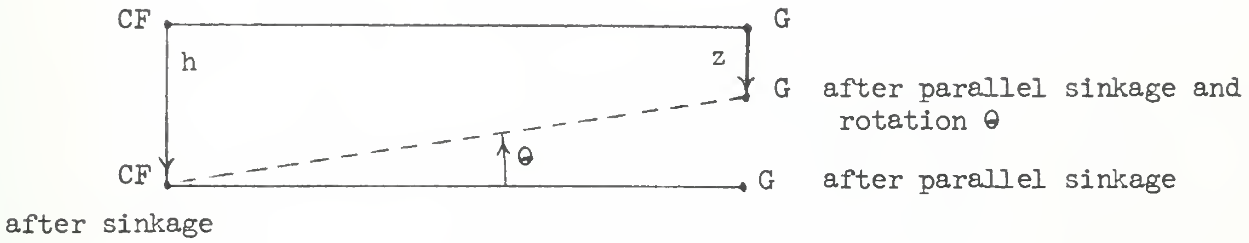
$k_h \equiv$ coefficient of Leave damping

T_{IB} = thrust available against ice

T_f = pounds per foot immersion

Figure B-X

Relationship of h to z and θ



CF = center of flotation on original ship waterline

G = center of gravity

Step 1. Sink ship h in parallel fashion.

Step 2. Trim about CF (which does not effect buoyance magnitude).

Note that LCF and/or LCG are negative if they are aft of amidships.

Therefore the radius of rotation is (LCG-LCF).

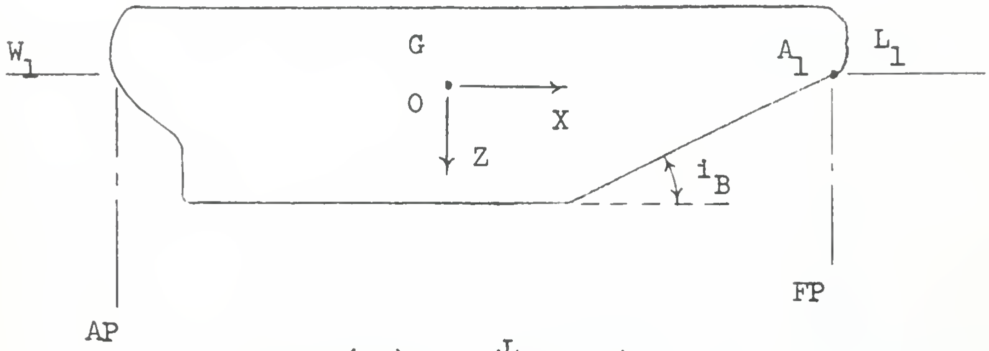
$$z = h - (LCG-LCF)\theta$$

or

$$h = z + (LCG-LCF)\theta \qquad (B17)$$

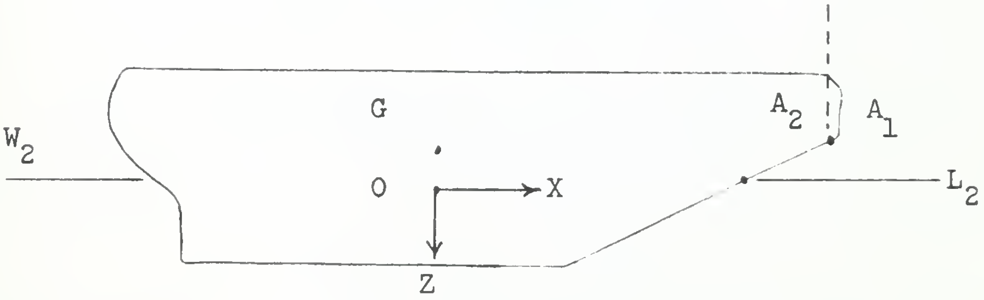
Figure B-XI

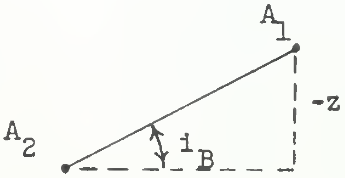
Change of $(GA)_x$ Caused by Change in z



Original $(GA)_x = (\frac{L}{2} - LCG)$

Raise ship (-z)



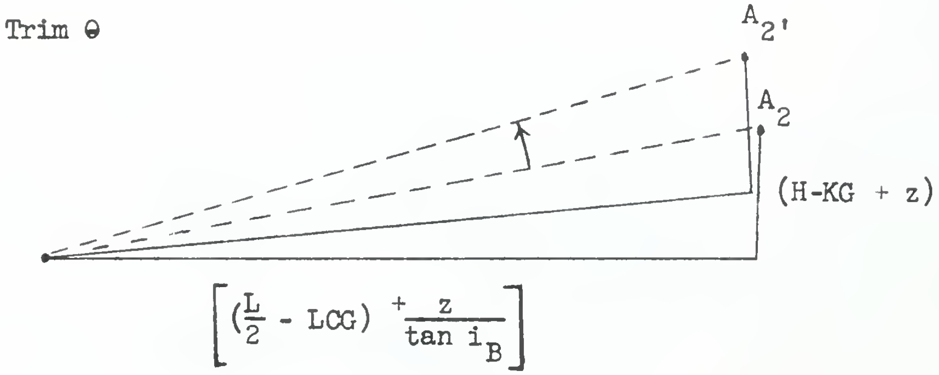


with ship raised,

$$(GA)_x = (\frac{L}{2} - LCG) \frac{-(-z)}{\tan i_B}$$

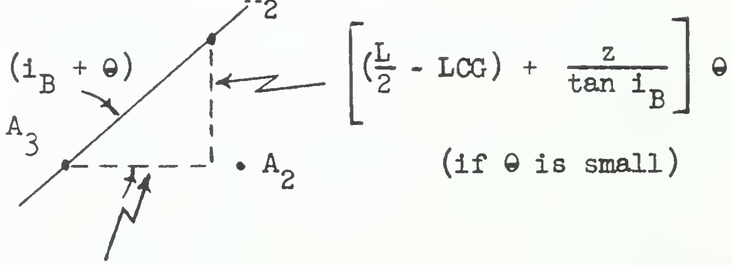
Figure B-XII

Change of $(GA)_x$ Caused by Change of Trim After Change of z.


Trim Θ
$A_{2'}$
A_2
$(H\text{-}KG + z)$
$\left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right]$

$$(GA)_x = \left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right] \cos \Theta - (H\text{-}KG + z) \sin \Theta$$

to 2'


A_2'
$(i_B + \Theta)$
A_3
A_2
$\left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right] \Theta$
(if Θ is small)

$$\frac{1}{\tan\,(i_B + \Theta)} \left[\left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B}\right] \Theta$$

$$\tan\,(i_B + \theta) = \frac{\sin i_B + \theta \cos i_B}{\cos i_B - \theta \sin i_B}$$

Set $\sin i_B = C$, and $\cos i_B = e$

Then $\dfrac{1}{\tan\,(i_B + \theta)} = \dfrac{e - \theta c}{c + \theta e} = \dfrac{e}{c} \left(\dfrac{1 - \frac{c}{e}\theta}{1 + \frac{e}{c}\theta} \right.$

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{e}{c} \left[1 - \left(\frac{c}{e} + \frac{e}{c}\right) \theta + \frac{e}{c} \left(\frac{c}{e} + \frac{e}{c}\right) \theta^2 - \ldots\ldots \right]$$

The term containing θ^2 is negligible and therefore

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{\cos i_B}{\sin i_B} \left[1 - \left(\frac{\sin i_B}{\cos i_B} + \frac{\cos i_B}{\sin i_B}\right) \theta \right]$$

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{1}{\tan i_B} - \theta - \frac{\theta}{\tan^2 i_B}$$

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{1}{\tan i_B} - \left(1 + \frac{1}{\tan^2 i_B}\right) \theta$$

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{1}{\tan i_B} - \frac{\theta}{\sin^2 i_B} \qquad \text{(B19)}$$

Substitute equation (B18) into (B19) and expand the equation.

$$(GA)_x = \left(\frac{L}{2} - LCG\right) + \frac{z}{\tan i_B} - (H - KG)\,\theta - z\,\theta$$

$$- \frac{\left(\frac{L}{2} - LCG\right)\theta}{\tan i_B} - \frac{z\,\theta}{\tan^2 i_B} + \frac{\left(\frac{L}{2} - LCG\right)\theta^2}{\sin^2 i_B}$$

$$+ \frac{z\,\theta^2}{\tan i_B \sin^2 i_B}$$

Considering the non-linear terms as negligible,

$$(GA)_x = (\frac{L}{2} - LCG) - \left[(H-KG) + \frac{(\frac{L}{2} - LCG)}{\tan i_B} \right] \theta$$

$$+ \left[\frac{-1}{\tan i_B} \right] z \qquad \text{(B20)}$$

The buoyance force, $(\Delta + T_f h)$, acts upward through the center of buoyance. When θ is positive B is aft of G by a distance of $GM_L \theta$, where GM_L is the height of the longitudinal metacenter above the center of gravity.

Icebreaking Thrust

It is assumed that the thrust available for icebreaking acts parallel to the base line at a height of (d) above the keel. Therefore, the lever arm for the $(T_{IB} \cos \theta)$ term is (KG - d). See Figure B-IX.

Thrust just prior to impact is utilized in overcoming "non-ice" resistance. At State 0,

$$T (1 - t) = R_T \qquad \text{(B21)}$$

where t is the thrust deduction factor, R_T is the total "non-ice" resistance, and T is the propeller thrust.

The thrust available for icebreaking, T_{IB}, may be defined as follows:

$$T_{IB} = T(1 - t) - R_T \qquad \text{(B22)}$$

It is noted that

$$R_T = (C_r + C_f) \rho /2 \; S \; v^2 \qquad \text{(B23)}$$

where C_r = Coefficient of residual resistance

C_f = Coefficient of frictional resistance

ρ = Density of water (constant)

S = Wetted surface area (constant)

v = Ship velocity

At low $\frac{V}{\sqrt{L}}$ values (below 0.5), C_r may be considered constant. (31)

Gebers indicates

$$C_f = 0.02058 \left(\frac{vL}{r}\right)^{-\frac{1}{8}}$$

while Prandtl and Von Karman indicate

$$C_f = 0.072 \left(\frac{vL}{v}\right)^{-\frac{1}{5}}$$

where L = Length of ship (constant)

v = Kinematic viscosity (constant)

It may be seen that

$$R_T = K_1 v^2 + K_2 v^{(2 - \frac{1}{8})}$$

(Using Gebers' equation). The first term is for residual resistance while the second is for frictional resistance.

$$R_T = K_1 v^2 + K_2 v^{15/16}$$

This equation may be written in the following approximate form:

$$R_T \cong K_3 v^2 \qquad \text{(B24)}$$

The total resistance (including ice) which may be opposed is

$$T\ (i - t)$$

The thrust deduction factor is virtually independent of v.

$$T = K_T\ \rho\ n^2\ d^4$$

where n = propeller revolutions per second

$\mathcal{D}$ = propeller diameter (constant)

It will be assumed that n remains constant throughout the crushing phase (and the sliding phase). (See Figure B XIII).

Using the Taylor wake fraction, a relationship between v_o (approach velocity to propeller) and v (ship velocity) may be set up (31).

$$v_o = v(1 - w)$$

where w = Taylor Wake Fraction

The wake fraction is virtually independent of ship velocity over most of the range.

As is illustrated in Figure XIV the thrust varies approximately linearly with ship velocity. Furthermore, since t is virtually constant,

$$T\,(1 - t) = K_4 - K_5\,v \qquad (B25)$$

It is to be noted that T (1 - t) at a ship velocity equal to zero this force is commonly known as "bollard pull", T_{BOL}.

Therefore, in equation (B25)

$$K_4 = T_{BOL}$$

and equation (B25) becomes

$$T\,(1 - t) = T_{BOL} - K_5\,v \qquad (B26)$$

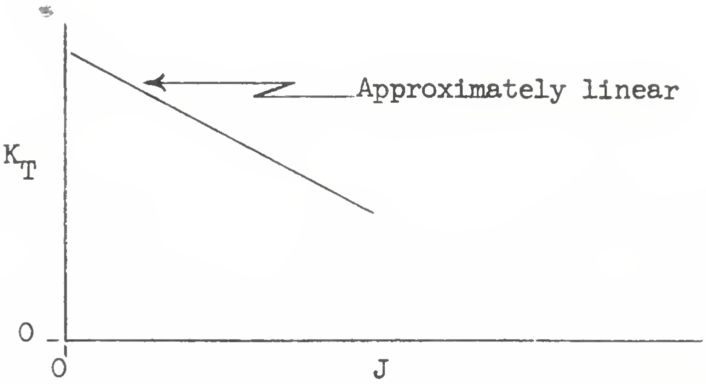
where v_1 = velocity of ship in the x-direction at State 1, equation (B26) becomes

$$T\,(1 - t) = T_{BOL} - K_5\,v_1$$

and from equation (B21) and equation (B24),

Figure B-XIII

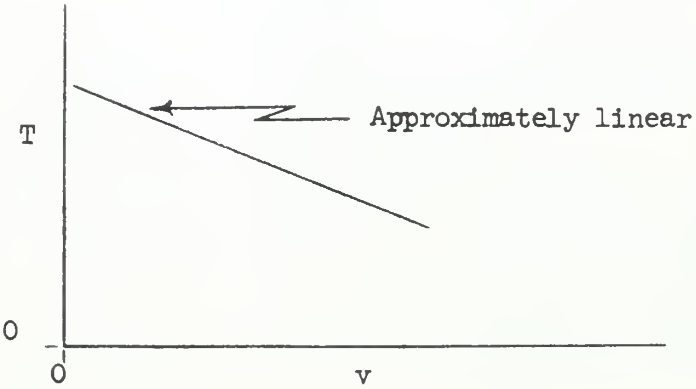
Illustration of Propeller Design Chart (31) Showing Variation of K_T with J.



where $J = \frac{v_o}{n\ d}$ (v_o = approach velocity to propeller)

Figure B-XIV

Variation of T with Ship Velocity



(n = constant)

$$R_T = T\,(1 - t)$$

$$K_3 v_1^2 = T_{BOL} - K_5 v_1$$

$$K_3 = \frac{T_{BOL} - K_5 v_1}{v_1^2}$$

From equation (B22)

$$T_{IB} = (T_{BOL} - K_5\ v) - \left(\frac{T_{BOL} - K_5\ v_1}{v_1^2}\right) v^2$$

$$T_{IB} = T_{BOL} - \left[K_5\right] v - \left[\frac{T_{BOL} - K_5 v_1}{v_1^2}\right] v^2 \qquad \text{(B27)}$$

Figure XV shows illustrative plots of equations (B24) and (B26). Figure XVI illustrates equation (B27).

Note that equation has an unknown constant, K_5. This can be solved for only by knowing the thrust and resistance characteristics for a wide range of impact speeds. These could only be known if other variables (i.e., ρ, n, d, S, ν, C_r, C_f etc.) were known and introduced. Furthermore, the equation is non-linear so it could not be used in linear differential equations even if K_5 were known.

In the crushing phase, a good linear approximation could be made by determining the slope of the curve at v_1. However, the slope is also a function of K_5 so in spite of the fact this would lead to a linear equation it would be unduly complex.

The next best approximation would be the one illustrated in Figure B-XVI.

Figure B-XV
Thrust and Resistance Forces vs Ship Speed

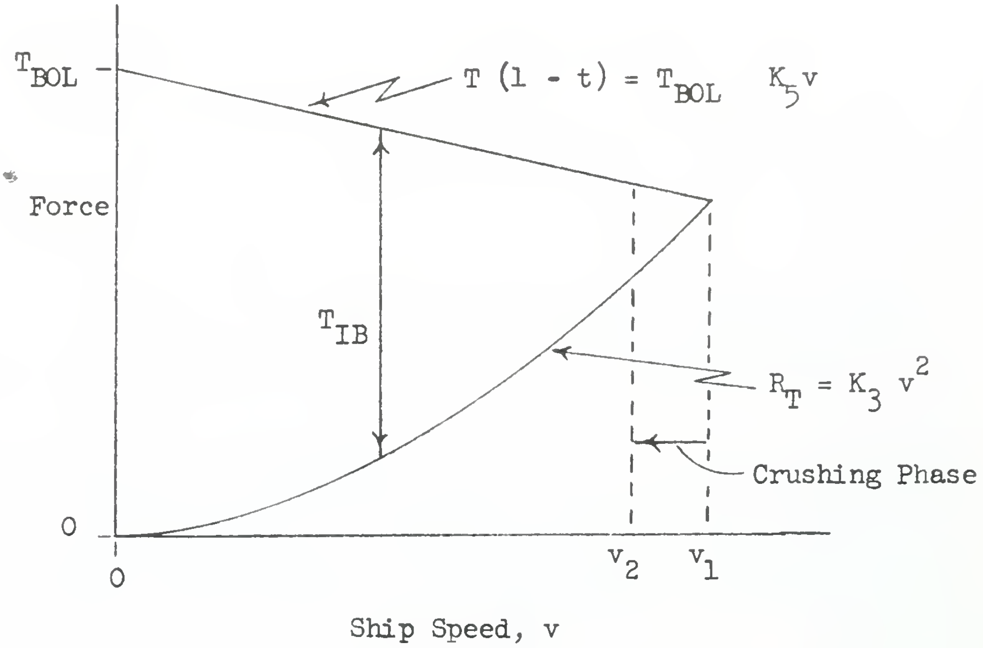
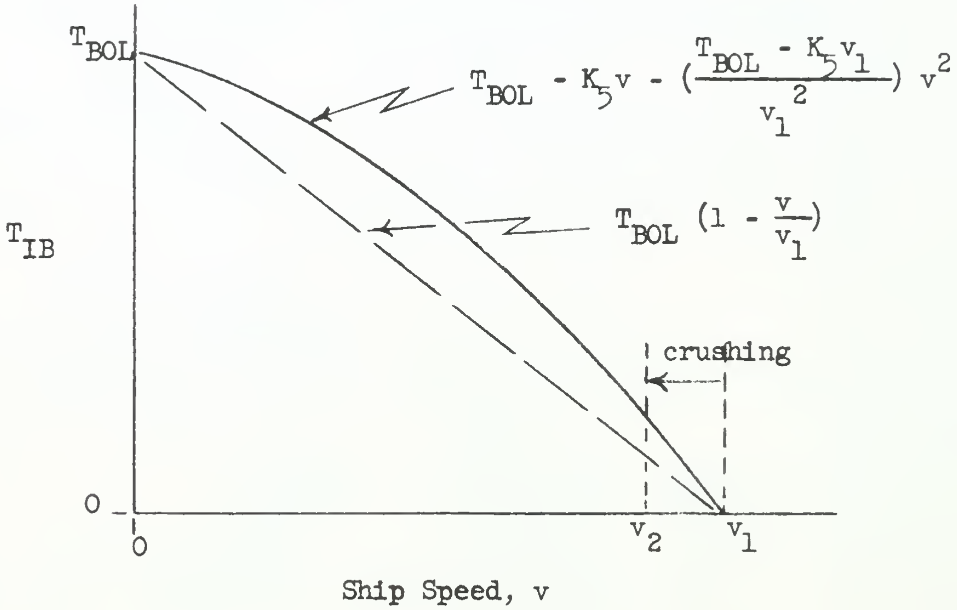


Figure B-XVI
T_{IB}, Icebreaking Thrust vs Ship Speed



$$T_{IB} = T_{BOL}\left(1 - \frac{v}{v_1}\right) \qquad (B28)$$

$$\text{where} \quad v = \frac{dx}{dt}$$

Incidentally, this approximation will be valid during the sliding phase as well as the crushing phase.

It should be reiterated that T_{BOL} is the "bollard pull" generated by using the same rpm that is necessary to maintain v_1 in open water.

Newton's Laws of Motion During Crushing

Newton's Laws of Motion may now be applied for the crushing phase rotationally about the center of gravity, in the x-direction, and in the z-direction.

In the horizontal direction (see Figure B-IX)

$$\sum F_x = m_x \frac{d^2x}{dt^2}$$

where m = Mass of ship plus "virtual" mass in the x-direction

$$T_{IB} \cos \Theta - F_{BXC} = m_x \frac{d^2x}{dt^2}$$

Setting $\cos \Theta = 1$ and substituting equations (B15) and (B28) this becomes

$$T_{BOL}\left(1 - \frac{v}{v_1}\right) - \left[\frac{\sigma \tan i_B}{\sin \beta} \sin i_B (\cos \beta + f_k \sin \beta) + \frac{\sigma \tan i_B}{\sin \beta} f_k \cos i_B\right] x^2 - m \frac{d^2x}{dt^2} = 0$$

$$- m \frac{d^2x}{dt^2} - \frac{T_{BOL}}{v_1} \frac{dx}{dt} - \frac{\sigma \tan i_B}{\sin \beta}\left[(\cos \beta + f_k \sin \beta) \sin i_B + f_k \cos i_B\right] x^2 + T_{BOL} = 0$$

Set

$$a_1 = \frac{T_{BOL}}{m_x v_1}$$

$$a_2 = \left[\frac{\sigma \tan i_B}{m_x \sin \beta} \quad (\cos \beta + f_k \sin \beta)\sin i_B + f_k \cos i_B\right]$$

$$a_3 = \frac{-T_{BOL}}{m_x}$$

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x^2 + a_z = 0 \qquad (B29)$$

Note that equation (B29) is independent of z and Θ and can therefore be solved as an independent equation.

$$\text{At } t_c = 0 \qquad x_1 = 0 \quad \left(\frac{dx}{dt}\right)_1 = v_1 \quad \left(\frac{d^2x}{dt^2}\right)_1 = 0$$

Note that the solution of equation (B29) is a function of t. As a consequence the solutions to equations (B14) and (B15), F_{BZC} and V_{BXC} respectively are functions of t.

$$F_{BZC} = f(t) \qquad F_{BXC} = f(t) \qquad (B30)$$

It can be seen in Figure B-IX in the downward vertical direction (z-direction) that

$$F_z = m_z \frac{d^2z}{dt^2}$$

$$\Delta - k_h \frac{dz}{dt} - (\Delta + T_f h) - T_{IB} \sin \Theta - F_{BZC}$$

$$= m_z \frac{d^2z}{dt^2}$$

Making appropriate substitutions from equations (E17), (E14), and (B28) this becomes

$$-k_h \frac{dz}{dt} - t_{fz}^{*} - T_f\ (LCG\text{-}LCF)\theta - T_{BOL}\ \theta + \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right) \theta$$

$$- \left[\frac{\sigma \tan i_B}{\sin \beta} \cos i_B\ (\cos \beta + f_k \sin \beta) - \frac{\sigma \tan i_B}{\sin \beta} f_k \sin i_B\right] x^2$$

$$- m_z \frac{d^2z}{dt^2} = 0$$

$$\left[- m_z\right] \frac{d^2z}{dt^2} + \left[- k_h\right] \frac{dz}{dt} + \left[- T_f\right] z$$

$$+ \left[- T_f\ (LCG - LCF) - T_{BOL} + \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\right] \theta$$

$$- \left[\cos i_B\ (\cos \beta + f_k \sin \beta) - f_k \sin i_B\right] \frac{\sigma \tan i_B}{\sin \beta}\ x^2 = 0 \qquad \text{(B31)}$$

It can be seen in Figure B-IX that the summation of moments about the center of gravity in the counterclockwise ($+\theta$) direction may be expressed as follows:

$$\sum M = m_\theta\, k^2 \frac{d^2\theta}{dt^2}$$

where k = radius of gyration.

$$F_{BZC}(GA)_x + F_{BXC}\ (GA)_z + T_{IB} \cos \theta\ (KG - d)$$

$$- (\Delta + T_f h) GM_L \theta - k_p \frac{d\theta}{dt} - m_\theta k^2 \frac{d^2\theta}{dt^2} = 0$$

Making appropriate substitutions from equations (B30), (B14), (B20), (B16), and (B28) this becomes

$$F_{BZC}(\frac{L}{2} - LCG) - F_{BZC}\left[(H-KG) + \frac{(L/2 - LCG)}{\tan i_B}\right]\theta$$

$$+ F_{BZC}\left[\frac{1}{\tan i_B}\right]z + F_{BXC}\left[H-KG\right] + F_{BXC}\left[z\right]$$

$$+ T_{BOL}\ (IG-d) - \left[\frac{T_{BOL}}{v_1}(KG-d)\right]\frac{dx}{dt}$$

$$- \left[\Delta\ GM_L\right]\theta - \left[T_f\ GM_L\right]\theta\ z - \left[T_f GM_L (LCG-LCF)\right]\theta^2$$

$$- k_p\frac{d\theta}{dt} - m\ k^2\frac{d^2\theta}{dt^2} = 0$$

In this equation both terms which include T_f are multiplied by non-linear terms. As shown earlier, these particular non-linear terms are minute compared to the other terms in the equation and will be dropped. The equation may now be written as follows:

$$\left[\frac{F_{BZC}}{\tan i_B} + F_{BXC}\right]z$$

$$+ \left[- m\ k^2\right]\frac{d^2\theta}{dt^2}$$

$$+ \left[- k_p\right]\frac{d\theta}{dt}$$

$$+ \left[-\Delta GM_L - F_{BZC}\ (H-KG) - F_{BZC}\ \frac{(\frac{L}{2} - LCG)}{\tan i_B}\right]\theta$$

$$+ \left[F_{BZC}\left(\frac{L}{2} - LCG\right) + F_{BXC}\,(H-KG) + T_{BOL}(KG-d) - \frac{T_{BOL}}{v_1}\,(KG-d)\,\frac{dx}{dt} \right] = 0 \qquad \text{(B32)}$$

Note that F_{BZC}, F_{BXC}, and $\frac{dx}{dt}$ are functions of t based on the solution of equation (B29).

Solution for x During Crushing

Many attempts, too many and too lengthy to be shown here, have been made to solve the non-linear summation equation (B29). However, it would seem that an assumption concerning the second and third terms is in order. It is noted that these are

$$\frac{T_{BOL}}{m\, v_1} \frac{dx}{dt} \quad \text{and} \quad \frac{-T_{BOL}}{m}$$

and combined are

$$\frac{T_{BOL}}{m} \left(\frac{\frac{dx}{dt}}{v_1} - 1 \right) .$$

This combined term is very small in the crushing range (See Figure B-XVI). In fact, it can readily be seen that the whole term is non-existent at initial contact.

Equation (B29) may now be written as

$$\sum F_x = m \frac{d^2x}{dt^2}$$

$$- F_{BXC} = m \frac{d^2x}{dt^2}$$

$$\text{Set } k_1 = \left[\frac{\sigma \tan i_B}{\sin \beta} \sin i_B (\cos \beta + f_k \sin \beta) + \frac{\sigma \tan i_B}{\sin \beta} f_k \cos i_B \right] \qquad \text{(B33)}$$

(See equation (B14))

Then $F_{BXC} = k_1 x^2$ (B34)

$$-k_1 x^2 = m \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = \frac{-k_1}{m} x^2 \qquad \text{(B35)}$$

Set $p = x' = \frac{dx}{dt}$ in accordance with reference (32)

$$\frac{d^2x}{dt^2} = \ddot{x} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dp}{dt} = \frac{dp}{dx}\ \frac{dx}{dt} = p\ \frac{dp}{dx}$$

(Note that p = velocity of the center of gravity in the x-direction)

$$p \frac{dp}{dx} = \frac{-k_1}{m} x^2$$

$$\int_{v_1}^{v} p\ dp = \frac{-k_1}{m} \int_0^x x^2 dx$$

$$\left[\frac{p^2}{2}\right]_{v_1}^{v} = \frac{-k_1}{m} \left[\frac{x^3}{3}\right]_0^x$$

$$\frac{p^2}{2} - \frac{v_1^2}{2} = \frac{-k_1}{3m}\ x^3$$

$$\frac{p^2}{2} = \frac{v_1^2}{2} - \frac{k_1}{3m}\ x^3$$

$$p = \left(v_1^2 - \frac{2k_1}{3m} x^3\right)^{1/2} = \frac{dx}{dt} \qquad \text{(B36)}$$

$$\int_0^x \left(v_1^2 \quad \frac{-2k_1}{3m} x^3\right)^{-\frac{1}{2}} cx = \int_0^t dt$$

$$\int_0^x \frac{dx}{\sqrt{\frac{-2k_1}{3m} x^3 + v_1^2}} = t \qquad \text{(B37)}$$

It is apparent that any exact solution to equation (B37) will be quite complex, unnecessarily complex.

Set $a = \frac{-2k_1}{3m}$ and $b = v_1^2$

temporarily.

Then the denominator of equation (B37) may be put into the numerator as

$$f(x) = (a\,x^3 + b)^{-\frac{1}{2}}$$

This function may be expanded into a series using Maclaurin's Theorem (33).

$$f(x) = f(0) + f'(0)\frac{x}{1!} \text{ x } f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

$$f(x) = (a\,x^3 + b)^{-\frac{1}{2}}$$

$$f'(x) = -\frac{1}{2}(a\,x^3 + b)^{-\frac{3}{2}}(3\,a\,x^2)$$

$$f''(x) = -\frac{1}{2}(a\,x^3 + b)^{-\frac{3}{2}}(b\,a\,x) + \frac{3}{2}(a\,x^3 + b)^{-\frac{5}{2}}(3a\,x^2)(3\,a\,x^2)$$

$f(0) = b^{-1/2}$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 0$ etc.

(Note: Taylor's Theorem may be used with values where zero is not used but the result, in effect, is approximately the same but more cumberson.)

Equation (B37) now becomes

$$\frac{1}{\sqrt{b}} \int_0^x dx = t$$

$$\frac{x}{v_1} = t \qquad x = v_1 t \qquad \text{(B38)}$$

In effect this states that velocity is approximated as constant. This value for x may now be substituted into equations (B35) and (B36) giving the following equations:

$$\frac{d^2x}{dt^2} = \left(\frac{-k_1}{m_x}\right) x^2 = \frac{-k_1}{m_x} v_1^2 t^2 \qquad \text{(B39)}$$

$$\frac{dx}{dt} = \left(v_1^2 - \frac{-2k_1}{3\,m_x} x^3\right)^{1/2} = \left(v_1^2 \quad \frac{-2k_1}{3\,m_x} v_1^3 t^3\right)^{1/2} \qquad \text{(B40)}$$

$$x = v_1 t$$

It is advisable to check the validity of equation (B38) by substituting appropriate values into equation (B38) and (B39) and seeing if $\frac{dx}{dt}$ drops off excessively, too excessively to use equation (B38) for x.

Assume the following approximate values (14):

$$m = \frac{17.28 \times 10^6}{32.2} = 5 \times 10^5 \ \frac{\text{lb sec}^2}{\text{ft}}$$

$$v_1 \cong 10 \text{ ft/sec}$$

$$\sigma \cong 4 \times 10^4 \quad \text{psf} \ (20 \text{ kg/cm}^2)$$

$i_B = 30^\circ$ $\tan i_B \cong 0.6$ $\sin i_B = 0.5$ $\cos i_B \cong 0.9$

$\beta = 40^\circ$ $\sin \beta = 0.6$ $\cos \beta \cong 0.8$

$f_k = 0.2$ $x \cong 10$ ft

From equation (B33),

$$k_1 = \frac{(4 \times 10^4)(0.6)}{(0.6)} \left[(0.5)\ (0.8 + 0.12) + (0.18) \right]$$

$$k_1 = (4 \times 10^4)\ (0.46 + 0.18) = (4 \times 10^4)\ (0.64)$$

$$k_1 \cong 2.6 \times 10^4 \text{ lb/ft}^2$$

$$\frac{k_1}{m} \cong \frac{2.6 \times 10^4}{5 \times 10^5} \cong 5 \times 10^{-2}$$

$$-\frac{2}{3}\ \frac{k_1}{m}\ v_1^3\ t^3 = -(\tfrac{2}{3})(5 \times 10^{-2})(10^3)(1) = -33$$

$$\left(\frac{dx}{dt}\right)_{x=10'} = (100 - 33)^{1/2} = (66)^{1/2} = 8.1 \text{ ft/sec}$$

If the velocity decay had been linear this would imply that the distance travelled in 1 sec would have been

$$\left(\frac{10 + 8.1}{2}\right) = 9.05 \text{ ft/sec}$$

$$(9.05)(1) = 9.05 \text{ ft}$$

instead of 10 ft. (Since acceleration is increasingly negative, the velocity decay would not have been linear and the distance travelled in 1 sec would have been even closer to the 10 ft we started with.)

Using a series expansion it is possible to expand f(x) of the integral of equation (B37) remembering that there exists the condition

$$(\tfrac{a}{b})^2 \; x^6 < 1$$

$$f(x) = b^{-1/2}\left[1 - \tfrac{1}{2}(\tfrac{a}{b})\, x^3 + \tfrac{1.3}{2.4}(\tfrac{a}{b})^2\, x^6 - \tfrac{1.3.5}{2.4.6}(\tfrac{a}{b})^3\, x^9 + \ldots .\right]$$

$$t = \int_0^x f(x)\, dx$$

$$t = \frac{1}{b^{1/2}} \int_0^x \left[1 - \tfrac{1}{2}(\tfrac{a}{b})\, x^3 + \tfrac{3}{8}(\tfrac{a}{b})^2\, x^6 - \tfrac{5}{8}(\tfrac{a}{b})^3\, x^9\right] dx$$

$$t = \frac{1}{b^{1/2}}\left[x - \tfrac{1}{8}(\tfrac{a}{b})\, x^4 + \tfrac{3}{56}(\tfrac{a}{b})^2\, x^7 - \tfrac{1}{16}(\tfrac{a}{b})^3\, x^{10}\right]$$

$$a = \frac{-2k_1}{3\ m_z} \qquad b = v_1^2$$

$$a = \frac{-2(2.6 \times 10^4)}{3(5 \times 10^5)} = -3.47 \times 10^{-2}$$

$$b = 10^2$$

Assume $x = 4$ ft

$(\frac{a}{b}) = -3.47 \times 10^{-4}$ $\qquad x^4 = 2.58 \times 10^2$

$(\frac{a}{b})^2 = +1.205 \times 10^{-7}$ $\qquad x^7 = 1.65 \times 10^4$

$(\frac{a}{b})^3 = -4.18 \times 10^{-11}$ $\qquad x^{10} = 1.02 \times 10^6$

$-\frac{1}{8}(\frac{a}{b})\, x^4 = +1.12 \times 10^{-2}$

$$+ \frac{3}{56} \left(\frac{a}{b}\right)^2 x^7 = - 1.068 \times 10^{-4}$$

$$- \frac{1}{16} \left(\frac{a}{b}\right)^3 x^{10} = + 2.56 \times 10^{-6}$$

$$t = \frac{1}{10} \left[4 + 0.0112 - 0.000107 + 0.0000026 \right]$$

$$t = \frac{4.0113096}{10} = 0.40113096 \text{ seconds}$$

If it had been assumed that

$$t = \frac{x}{v_1} \qquad t = \frac{4}{10} = 0.40 \text{ seconds.}$$

Certainly it is satisfactory to use this relationship where the assumptions prove themselves correct later. This confirms equation (B38).

Using only the first and second terms,

$$t = \frac{1}{b^{1/2}} \left[x - \frac{1}{8} \left(\frac{a}{b}\right) x^4 \right]$$

$$b^{1/2} \; t = x - \frac{a}{8b} \; x^4 \qquad \text{where } a = \frac{-2k_1}{3 \; m_x} \quad b = v_1^2$$

By using a reversion of the series (34)

$$(b^{1/2}t) = (1)x + (0)x^2 + (0)x^3 + \left(\frac{-a}{8b}\right) x^4$$

the equation becomes

$$x = A_1(b^{1/2}) + A_2(b^{1/2}t) + A_3(b^{1/2}t)^2 + A_4(b^{1/2}t)^3 + \ldots$$

$$A_1 = \frac{1}{1} = 1 \qquad A_2 = 0 \qquad A_3 = 0$$

$$A_4 = \frac{1}{1} \left(+ \frac{a}{8b}\right) \qquad A_5 = 0 \text{ etc.}$$

$$x = b^{1/2}\, t \qquad + \overset{3/2}{\frac{ab}{8b}}\, t^3$$

$$x = v_1 t \quad \frac{-2k_1 v_1^3}{24\, m_x v_1^2}\, t^3 \qquad x = v_1\, (t - \frac{k_1 t^3}{12\, m_x}) \tag{B41}$$

This equation, (B41), will be used.

Then

$$\dot{x} = (v_1^2 \quad \frac{-2k_1 x^3}{3\, mx})^{1/2} \tag{B42}$$

$$\ddot{x} = (\frac{-k_1}{m_x})\, x^2 \tag{B43}$$

These three equations will be used to express x, $\dot{x}$, and $\ddot{x}$ but since the last term of equation (B41) is almost always negligible it is dropped in calculations of other coordinates.

Solution for θ During Crushing

The equation for the summation of the forces in the z-direction during crushing, (B31), may now be rewritten. The term for T_{IB} will be dropped for the reasons mentioned earlier.

Let

$$k_2 = \frac{\sigma \tan i_B}{\sin \beta} \left[\cos i_B (\cos \beta + f_k \sin \beta) - f_k \sin i_B \right] \qquad \text{(B44)}$$

$$\left[-m_x\right] \frac{d^2 z}{dt^2} + \left[-k_h\right] \frac{dz}{dt} + \left[-T_f\right] z$$

$$+ \left[-T_f(\text{LCG-LCF})\right] \theta$$

$$\left[-k_2\right] x^2 = 0 \qquad \text{(B45)}$$

$$x^2 = v_1^2 t^2$$

Let $a_{12c} = + m_z$ $\quad b_{12c} = + k_h$ $\quad c_{12c} = + T_f$

$a_{13c} = 0$ $\quad b_{13c} = 0$ $\quad c_{13c} = + T_f\ (\text{LCG-LCF})$

(Note: These are constant coefficients)

Then

$$a_{12c}\ \ddot{z} + b_{12c}\ \dot{z} + c_{12c}\ z,$$

$$+ a_{13c}\ \ddot{\theta} + b_{13c}\ \dot{\theta} + c_{13c}\ \theta = k_2 v_1^2\ t^2 \qquad \text{(B46)}$$

The equation for the summation of the moments about the center of gravity during crushing, (B32), may now be rewritten. The term for T_{IB}

will be dropped for reasons mentioned earlier.

In order to keep the coefficients of the equation constant, the first term,

$$\left[\frac{F_{BZC}}{\tan i_B} + F_{BXC}\right] z$$

will be dropped. This simplification should be valid since the magnitude of the term is negligible when compared to the term

$$\left[F_{BZC}\left(\frac{L}{2} - LCG\right) + F_{BXC}\,(H-KG)\right]$$

which will be retained. It is noted that this latter term is in the order of 200 times larger than the former.

$$\left[-m\,k^2\right]\frac{d^2\theta}{dt^2} + \left[-k_p\right]\frac{d\theta}{dt}$$

$$+ \left[-\Delta GM_L - F_{BZC}\,(H-KG) - F_{BZC}\,\frac{\left(\frac{L}{2} - LCG\right)}{\tan i_B}\right]\theta$$

$$+ \left[F_{BZC}\left(\frac{L}{2} - LCG\right) + F_{BXC}\,(H-KG)\right] = 0$$

From equations (B14) and (B39) we find

$$F_{BZC} = k_2\,x^2 \qquad \text{(B47)}$$

From equations (B15) and (B33) we find

$$F_{BXC} = k_1\,x^2 \qquad \text{(B48)}$$

Substitution of equation (B38) leads to

$$F_{BZC} = k_2 \, v_1^2 \, t^2 \qquad \text{(B49)}$$

$$F_{BXC} = k_1 \, v_1^2 \, t^2 \qquad \text{(B50)}$$

The summation-of-moments equation may now be written

$$\left[+ mk^2\right] \frac{d^2\theta}{dt^2} + \quad + \left[k_p\right] \frac{d\theta}{dt}$$

$$+ \left[+ \Delta \, GM_L + k_2 v_1^2 \, (H\text{-}KG) t^2 + \frac{k_2 v_1^2 \left(\frac{L}{2} - LCG\right) t^2}{\tan i_B}\right] \theta$$

$$= \left[k_2 \, v_1^2 \left(\frac{L}{2} - LCG\right)\right] t^2 + \left[k_1 v_1^2 \, (H\text{-}KG)\right] t^2$$

Let $a_{23c} = \left[m_\theta k^2\right] \qquad b_{23c} = +\left[k_p\right]$

$$c_{23c} = \Delta GM_L + \left\{ k_2 v_1^2 \left[(H\text{-}KG) + \frac{\left(\frac{L}{2} - LCG\right)}{\tan i_B} \right] \right\} t^2$$

(Note that c_{23c} is a function of t.)

$$d_{23c} = v_1^2 \left[k_2 \left(\frac{L}{2} - LCG\right) + k_1 \, (H\text{-}KG) \right]$$

The summation of moments equation may now be written

$$a_{23c} \, \ddot{\theta} + b_{23c} \, \dot{\theta} + c_{23c} \, \theta = d_{23c} \, t^2 \qquad \text{(B51)}$$

Let $c = \Delta GM_L$, the constant portion of the factor of θ in equation (B51). Then this equation becomes

$$a\ddot{\theta} + b\dot{\theta} + c\theta = dt^2$$

$$\ddot{\theta} + \frac{b}{a}\dot{\theta} + \frac{c}{a}\theta = \frac{d}{a}t^2 \qquad \text{(B52)}$$

The related homogeneous equation is

$$\ddot{\theta} + \frac{b}{a}\dot{\theta} + \frac{c}{a}\theta = 0$$

Roots

$$r = \frac{-\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - 4\frac{c}{a}}}{2}$$

$$\theta = A_1 e^{\left(-\frac{b}{2a} + \frac{1}{2}\sqrt{\frac{b^2}{a^2} - 4\frac{c}{a}}\right)t} + A_2 e^{\left(-\frac{b}{2a} - \frac{1}{2}\sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}\right)t}$$

$$\theta = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad \text{Note: } \frac{b^2}{a^2} > \frac{4c}{a}$$

$$\frac{b^2}{a} > 4c$$

As seen before, the partial solution

Assume the partial solution to be

$$\theta_p = A t^3 + B t^2 + C t + D$$

$$\dot{\theta}_p = 3 A t^2 + 2 B t + C$$

$$\ddot{\theta}_p = 6 A t + 2 B$$

Substitute these values in equation (B52).

$$6 A t + 2 B + \frac{b}{a} 3A t^2 + \frac{b}{a} 2 Bt + \frac{b}{a} C$$

$$+ \frac{c}{a} A\ t^3 + \frac{c}{a} B\ t^2 + \frac{c}{a} C\ t + \frac{c}{a} D - \frac{d}{a} t^2 = 0$$

$$(3A \frac{b}{a} + B\frac{c}{a} - \frac{d}{a})\ t^2 + (6A + 2\frac{b}{a} B + \frac{cC}{a})\ t$$

$$+ (2B + \frac{b}{a} C + \frac{c}{a} D) + (\frac{c}{a} A)\ t^3 = 0$$

$$A = 0 \qquad B = \frac{d}{c} \qquad C = (\frac{-2bB}{a})\ (\frac{a}{c})$$

$$C = \frac{-2bB}{c} = \frac{-2bd}{c^2}$$

$$\frac{c}{a} D = - \frac{b}{a} C - 2B \qquad D = \frac{-b}{c} C \quad \frac{-2aB}{c}$$

$$D = (\frac{-b}{c})\ (\frac{-2bd}{c^2}) - (\frac{2a}{c})\ (\frac{d}{c})$$

$$D = \frac{2b^2 d}{c^3} - \frac{2ad}{c^2}$$

$$\theta_p = \frac{d}{c} t^2 + (\frac{-2bd}{c^2})\ t + \frac{2b^2 d}{c^3} - \frac{2\ ad}{c^2}$$

Find the partial solution, θ_p, of equation (B52) $R = \frac{d}{a} t^2 = P(t)$

$$P'(t) = \frac{2d}{a} t \qquad P''(t) = \frac{2d}{a}$$

$$\theta_p = \frac{1}{c/a} \left[\frac{d}{a} t^2 - \frac{b/a}{c/a}\ (2)\ \frac{d}{a} t + \frac{\frac{b^2}{a^2} - \frac{c}{a}}{c^2/a^2}\ (2)\ \frac{d}{a} \right]$$

$$\Theta_p = \left[\frac{a}{c} \; \frac{d}{a} t^2 - \frac{2\ bd}{ca} t + \frac{2\ d\ b^2 a^2}{a^3 c^2} - \frac{2\ cd\ a^2}{a^2 c^2} \right]$$

$$\Theta_p = \frac{d}{c} t^2 - \frac{2\ bd}{c^2} t + \frac{2\ b^2 d}{c^3} - \frac{2\ ad}{c^2}$$

This confirms previous solution of Θ_p.

$$\Theta_p = \frac{-2d}{c^2} \left[\left(\frac{b}{r_1} - a + \frac{b^2}{c} \right) \left(\frac{r}{r_2 - r_1} \right) + \frac{b^2}{c} - a \right] e^{r_1 t}$$

$$+ \left[\frac{2d}{c^2} \left(\frac{b}{r_1} - a + \frac{b^2}{c} \right) \left(\frac{r}{r_2 - r_1} \right) \right] e^{r_2 t}$$

$$+ \frac{d}{c} t^2 - \frac{2\ bd\ t}{c^2} + \frac{2\ b^2 d}{c^3} - \frac{2\ ad}{c^2}$$

where $(k_p^2 > 4\ m\ k^2\ \Delta\ GM_L)$

Before proceeding with $(b^2 \quad 4\ ac)$, look at solution where $b^2 \quad 4\ ac$ or equivalently where

$$k_p^2 < 4\ m\ k^2\ \Delta\ GM_L\ .$$

$$\ddot{\theta} + \frac{b_1}{a_1} \dot{\theta} + \frac{c_1}{a_1} \theta = \frac{d_1}{a_1} t^2 \qquad (B53)$$

$$r_1 = \frac{-b_1}{2a_1} + \frac{1}{2} \sqrt{\frac{b_1^2}{a_1^2} - \frac{4c_1}{a_1}} \qquad r_2 = \frac{b_1}{2a_1} - \frac{1}{2} \sqrt{\frac{b_1^2}{a_1^2} - \frac{4c_1}{a_1}}$$

(for the homogeneous equation) (Both roots contain imaginary terms.)

Let $\alpha_1 = \frac{-b_1}{2\, a_1}$ $\qquad \beta_1 = + \frac{1}{2}\, i \sqrt{\frac{b_1^2}{a_1^2} - \frac{4c_1}{a_1}}$

$$\theta = e^{\alpha_1 t} \quad (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t)$$

(General solution to the homogeneous equation)

As before, if $\theta_p = A_1 t^3 + B_1 t^2 + C_1 t + D_1$

the general solution of the complete nonhomogeneous equation becomes

$$\theta = e^{\alpha_1 t} \quad (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) + \frac{d_1}{c_1} t^2 - \frac{2\, b_1 d_1}{c_1^2} t + \frac{2\, b_1^2\, d_1}{c_1^3} - \frac{2\, a_1 d_1}{c_1^2} \qquad (B54)$$

$$t = 0 \quad \theta = 0 \_ A_1 + \frac{2\, b_1^2 d_1}{c_1^3} - \frac{2\, a_1 d_1}{c_1^2}$$

$$A_1 = \frac{2\, d_1}{c_1^2} \left(a_1 - \frac{b_1^2}{c_1}\right)$$

$$\dot{\theta} = \alpha_1\, e^{\alpha_1 t} \quad (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) + e^{\alpha_1 t} \quad (-A_1 \beta_1 \sin \beta_2 t + A_2 \beta_1 \cos \beta_1 t)$$

$$+ \frac{2\, d_1\, t}{c_1} - \frac{2\, b_1 d}{c_1^2} \qquad (B55)$$

$$t = 0 \quad \dot{\theta} = 0 = \alpha_1 (A_1) + (A_2 \beta_1) \quad \frac{-2\, b_1 d_1}{c_1^2}$$

$$A_2 = + \frac{2\, b_1 d_1}{c_1^2\, \beta_1} \quad \frac{-\alpha_1 A_1}{\beta_1} = \frac{2\, d_1}{c_1^2} \left[\left(\frac{b_1}{\beta_1} - \frac{\alpha_1}{\beta_1} (a_1 \; \frac{-b_1^2}{c_1}) \right) \right]$$

$$A_2 = \frac{2\, d_1}{c_1^2} \left[\frac{b_1 - \alpha_1 (a_1 - \frac{b_1^2}{c_1})}{\beta_1} \right]$$

The following terms are used concerning rotation during the crushing phase:

$$a_1 = m_\theta\, k^2 \qquad (B56)$$

$$b_1 = k_p$$

$$c_1 = \Delta\, GM_L$$

$$k_3 = \left[k_2 (\tfrac{L}{2} - LCG) + k_1 (H - KG) \right]$$

$$d_1 = v_1^2\, k_3$$

$$\alpha_1 = \frac{-b_1}{2a_1} \qquad (B57)$$

$$\beta_1 = + \frac{1}{2}\, i \sqrt{\frac{b_1^2}{a_1^2} - \frac{4c_1}{a_1}} = + \frac{1}{2} \sqrt{\frac{4c_1}{a_1} - \frac{b_1^2}{a_1^2}}$$

$$A_1 = \frac{2d_1}{c_1^2} (a_1 - \frac{b_1^2}{c_1})$$

$$A_2 = \frac{2d_1}{\beta_1 c_1^2} \left[b_1 - \alpha_1 (a_1 - \frac{b_1^2}{c_1} \right]$$

$$A_2 = \frac{2\ bd}{c^2\beta} - \frac{\alpha A_1}{\beta} = \frac{2\ bd}{c^2\beta} + \frac{b\ \frac{2d}{c^2}}{2\ a\ \beta}\left(a\ \frac{-b^2}{c}\right)$$

$$\frac{2\ bd}{c^2\beta} - \frac{\alpha}{\beta}\ \frac{2d}{c^2}\left(a - \frac{b^2}{c}\right) = \frac{2d}{c^2\beta}\left(b - \alpha\left(a - \frac{b^2}{c}\right)\right)$$

$$A_2 = \frac{2d}{c^2\beta}\left(b + \frac{b}{2a}\left(a - \frac{b^2}{c}\right)\right) = \frac{2d}{c^2\beta}\left(b + \frac{b}{2} - \frac{b^3}{2ac}\right)$$

$$A_2 = + \frac{2\ bd}{c^2\beta}\left(\frac{3}{2} - \frac{b^2}{2\ ac}\right) = \frac{bd}{c^2\beta}\left(3 - \frac{b^2}{ac}\right)$$

Equation (B54) may be written as

$$\theta = e^{\alpha_1 t}\left(A_1 \cos \beta_1 t + A_2 \sin \beta_1 t\right)\ \frac{+ d_1}{c_1} t^2 - \frac{2\ b_1 d_1}{c_1^2} t - A_1 \qquad \text{(B58)}$$

Check values of A_1 and A_2

$$\alpha_1 A_1 + A_2 \beta_1 \quad \frac{-2\ b_1 d_1}{c_1^2} = 0$$

$$\frac{2\ \alpha_1 d_1}{c_1^2}\left(a_1\ \frac{-b_1^2}{c_1}\right) + \frac{2\ b_1 d_1}{c_1^2} - \frac{2\ \alpha_1 d_1}{c_1^2}\left(a_1 - \frac{b_1^2}{c_1}\right) - \frac{2\ b_1 d_1}{c_1^2} = 0$$

A_1 and A_2 satisfy $\dot{\theta}$ and θ when $t = 0$.

$$\ddot{\theta} = \alpha_1^2 e^{\alpha_1 t} (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) - \beta_1^2 e^{\alpha_1 t} (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t)$$

$$+ 2 \alpha_1 \beta_1 e^{\alpha_1 t} (-A_1 \sin \beta_1 t + A_2 \cos \beta_1 t) + \frac{2 d_1}{c_1}$$

$$\ddot{\theta} = (\alpha_1^2 - \beta_1^2) e^{\alpha_1 t} (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) \tag{B59}$$

$$+ 2 \alpha_1 \beta_1 e^{\alpha_1 t} (-A_1 \sin \beta_1 t + A_2 \cos \beta_1 t) + \frac{2 d_1}{c_1}$$

It is to be noted that the homogeneous equation expressed damped oscillatory motion, which is what the ship would have if there were no moment applied. Therefore, equations (B54) and (B55), and (B59) will be used. Furthermore it may be anticipated that the $e^{\alpha t}$ term should be very close to unity at the low values of t we expect.

Test to see that $\ddot{\theta} = 0$ when $t = 0$.

$$\ddot{\theta}_{t=0} = A_1 (\alpha^2 - \beta^2) + 2 \alpha \beta A_2 + 2\frac{d}{c}$$

$$\ddot{\theta}_{t=0} = \frac{2d}{c^2} (a - \frac{b^2}{c}) \left[\frac{b^2}{4a^2} - \frac{1}{4} (\frac{4c}{a} - \frac{b^2}{a^2}) \right]$$

$$\frac{-b}{a} \quad \frac{bd}{c^2} (3 \quad \frac{-b^2}{ac}) + \frac{2d}{c}$$

$$\ddot{\theta} = \frac{2\,b^2 d}{4\,a^2 c^2}\left(a - \frac{b^2}{c}\right) - \frac{8\,cd}{4\,ac^2}\left(a - \frac{b^2}{c}\right) + \frac{2\,b^2 d}{4\,a^2 c^2}\left(a - \frac{b^2}{c}\right)$$

$$-\frac{3\,b^2 d}{ac^2} + \frac{b^4 d}{a^2 c^3} + \frac{2d}{c}$$

$$\ddot{\theta} = \frac{b^2 d}{2\,a\,c^2} - \frac{b^4 d}{2\,a^2 c^3} - \frac{2\,d}{c^2} + \frac{2\,b^2\,d}{a\,c^2} + \frac{b^2 d}{2\,a\,c^2}$$

$$- \frac{b^4 d}{2\,a^2 c^3} - \frac{3\,b^2 d}{ac^2} + \frac{b^4 d}{a^2 c^3} + \frac{2d}{c}$$

$$\ddot{\theta}_{t=0} = 0$$

Solution for z During Crushing

In equation (B45) the factor of θ will be considered negligible; it is a small corrective term for the difference of draft between the center of gravity and the center of flotation. The equation becomes

$$(+m_z)\ddot{z} + (k_h)\dot{z} + (T_f)z = (-k_2 v_1^2)\, t^2$$

$$a_2 = +m_z \qquad b_2 = k_h \qquad c_2 = T_f \qquad d_2 = (-k_2 v_1^2)$$

$$\ddot{z} + \frac{b_2}{a_2}\dot{z} + \frac{c_2}{a_2} z = \frac{d_2}{a_2} t^2 \tag{B60}$$

It is noted that equation (B60) is the same nonhomogeneous equation as (B53) and further that the motion represented by the homogeneous portion is the same type (damped oscillatory) that is represented by the solution to (B53). The initial conditions are the same.

We can therefore write the equations of motion using the following terms:

$$a_2 = +m_z \qquad b_2 = k_h \qquad c_2 = T_f \qquad d_2 = (-k_2 v_1^2) \tag{B61}$$

m = total mass

k_h = leave damping coefficient

T_f = lb/ft immersion

$$\alpha_2 = \frac{-b_2}{2\,a_2} \qquad \beta_2 = +\frac{1}{2}\,i\sqrt{\frac{b_2^2}{a_2^2} - \frac{4\,c_2}{a_2}} = \frac{1}{2}\sqrt{\frac{4\,c_2}{a_2} - \frac{b_2^2}{a_2^2}}$$

$$B_1 = \frac{2\,d_2}{c_2^2}\left(a_2 - \frac{b_2^2}{c_2}\right.$$

$$B_2 = \frac{2\,d_2}{c_2^2}\left[\frac{b_2 - \alpha_2\left(a_2 \;\; \frac{-b_2^2}{c_2}\right)}{\beta_2}\right]$$

The equations for motion during crushing in the vertical direction are

$$z = e^{\alpha_2 t}\left(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t\right) + \frac{d_2}{c_2} t^2$$

$$- \frac{2\,b_2 d_2 t}{c_2^2} + \frac{2\,b_2^2\,d_2}{c_2^3} - \frac{2\,a_2 d_2}{c_2^2} \qquad \text{(B62)}$$

$$\dot{z} = \alpha_2 e^{\alpha_2 t}\left(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t\right)$$

$$+ e^{\alpha_2 t}\left(-B_1 \beta_2 \sin \beta_2 t + B_2 \beta_2 \cos \beta_2 t\right)$$

$$\frac{+\,2\,d_2 t}{c_2} \quad \frac{-2\,b_2 d_2}{c_2^2} \qquad \text{(B63)}$$

$$\ddot{z} = \alpha_2^2\, e^{\alpha_2 t}\left(B_1 \cos \beta_2 t + B_2 \sin \beta_2 t\right)$$

$$+ 2\,\alpha_2 e^{\alpha_2 t}\left(-B_1 \beta_2 \sin \beta_2 t + B_2 \beta_2 \cos \beta\, t\right)$$

$$+ e^{\alpha_2 t}\left(-B_1 \beta_2^2 \cos \beta_2 t - B_2\, \beta_2^2 \sin \beta_2 t\right)$$

$$+ \frac{2\,d_2}{c_2}$$

$$\ddot{z} = (\alpha_2^2 - \beta_2^2)\, e^{\alpha_2 t} \left[B_1 \cos \beta_2 t + B_2 \sin \beta_2 t \right] \qquad (B64)$$

$$+ 2\, \alpha_2 \beta_2\, e^{\alpha_2 t} (-B_1 \sin \beta_2 t + B_2 \cos \beta_2) + \frac{2\, d_2}{c_2}$$

Velocity at the Bow

It can be seen that the velocity at point A on the bow must equal v_1, the approach velocity, at the time of initial contact. See Figure B-XVII.

As the criterion for the termination of the crushing phase, there must be no velocity at point A which is normal to the stem. Therefore the velocity of point A must be in the direction as indicated by $(i_B + \theta)$. See Figure B-XVIII.

Let the direction of the velocity of A at any time be defined by γ, as shown in Figure B-XIX. It follows that

$$\tan \gamma = \frac{(GA)_x \frac{d\theta}{dt} - \frac{dz}{dt}}{\frac{dx}{dt} - (GA)_z(\frac{d\theta}{dt})} \qquad (B65)$$

where $\frac{d\theta}{dt}$ is defined by equation (B55)

$\frac{dz}{dt}$ is defined by equation (B63)

and $\frac{dx}{dt}$ is defined by equation (B46)

We recall from equation (B16) that

$$(GA)_z = H\text{-}KG + z$$

and from equation (B20) that

$$(GA)_x = (\frac{L}{2} - LCG) - \left[(H\text{-}KG) + \frac{(\frac{L}{2} - LCG)}{\tan i_B} \right] \theta + (\frac{1}{\tan i_B}) z$$

Figure B-XVII

Point on Bow at First Contact with Ice. State 1

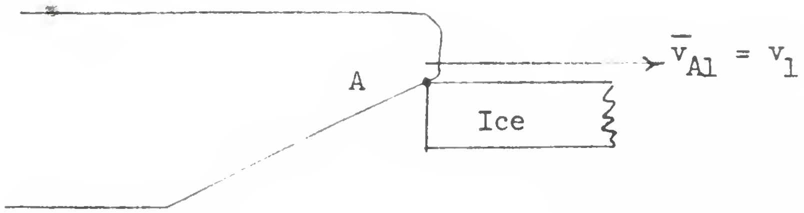


Figure B-XVIII

Point on Bow at End of Crushing Phase. State 2

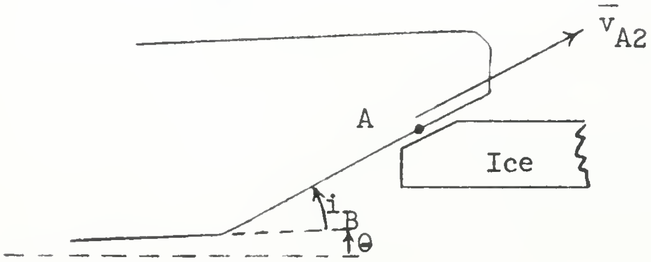
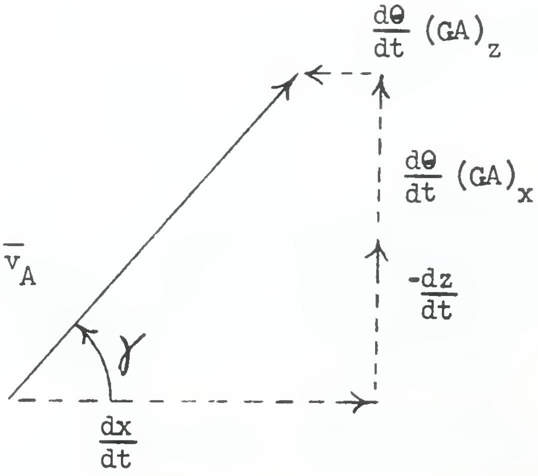


Figure B-XIX

Illustration of Vector Velocity of Point A on Bow

$\overline{v}_A = \overline{v}_G \quad \overline{v}_{A/G}$



Therefore, the equations on page 278 must also be solved on the basis of equations (B54) and (B62) before they may be substituted into equation (B65). If t is kept as the only unknown, then

$$\tan \gamma = f(t)$$

This may be solved for successive values of t until

$$\tan \gamma = \tan (i_B + \theta).$$

At this time, t_2, State 2 is reached and the crushing has stopped. Using t_2, any crushing equation of motion may then be solved.

The following values must be known:

f_k	Dimensionless coefficient of kinetic friction
β	Angle between normal to plating and centerline plane.
i_B	Angle between stem line and base line
σ	Estimate of compressive failure stress of ice, lbs. per square foot.
L	Length between perpendiculars, ft.
LCG	Distance from midships to center of gravity, + if forward, - if aft, ft.
H	Draft, ft.
KG	Height of center of gravity above keel, ft.
T_f	Pounds per foot immersion
k_p	Coefficient of pitch damping, ft-lb-sec
k_h	Coefficient of heave damping, lb-sec/ft

m_x	Mass of ship plus virtual mass in x-direction, $\frac{\text{lb-sec}^2}{\text{t}}$
m_z	Mass of ship plus virtual mass in z-direction, $\frac{\text{lb-sec}^2}{\text{ft}}$
k	Radians of gyration, ft
m_θ	Mass of ship plus virtual mass during rotation (pitch), $\frac{\text{lb-sec}^2}{\text{ft}}$
LCF	Distance from midships to center of flotation (+ if forward, - if aft), ft.
v_1	Velocity of ship immediately prior to initial contact, ft/sec
GM_L	Longitudinal metacentric height, ft.
Δ	Displacement in lb.
LCG	Distance from midships to center of gravity (+ if forward, - if aft), ft.

Mass and Mass Moment of Inertia

For the purpose of these calculations it may be assumed that the underwater shape of most icebreakers may be approximated as indicated in Table 62a on p. 423 of reference (24).

The following dimensions from a "Wind Class" Icebreaker will be used:

$$\text{LBP} = L = 250'$$

$$H = 25'9''$$

$$B = 62'0''$$

$$\Delta = 3500 \text{ tons}$$

$$\frac{L}{H} = \frac{250}{25.75} = 0.97$$

$$\text{Fatness Ratio} = \frac{(3500)(35)}{(25)^3} = 7.75$$

D "Maximum Diameter"

$$\frac{L}{D} = \frac{250}{62} = 4.03$$

Body No.	$\frac{L}{D}$	Fatness Ratio	CAMX	CAMZ	CAMθ
1	4	31.91	0.087	0.854	0.598
5	8	7.98	0.031	0.942	0.835

where CAMX is the added mass coefficient for unsteady motion along the x-axis.

CAMZ is the added mass coefficient for unsteady motion along the z-axis.

CAMθ is the added moment of inertia coefficient for pitch,

It is noted that in the reference the value for CAMZ is based on lateral translation but since these terms were developed for submerged shapes, the value is also valid for the z-direction.

Since none of the bodies in the table seem close enough to our typical shape, let us use the development of the prolate ellipsoid in Figure 62.B of the reference. This shape is not too far from the underwater shape of most icebreakers.

$a = \frac{L}{2} = 125 \qquad b = \frac{B}{2} = 31$

$a/b = 4.03$

For $\frac{a}{b} = 3.99 \qquad k_1 = 0.860 \qquad k_2 = 0.082 \qquad k_3 = 0.608$

where $k = \dfrac{\text{added mass (or mass moment of inertia)}}{\text{body mass (or mass moment of inertia)}}$

k_1 for the z-axis

k_2 for the x-axis

k_3 for pitching

Therefore, the mass (or mass moment of inertia) for ships of typical polar icebreaker form may be approximated as follows:

$$m_x = 1.08 \frac{\Delta}{g} = 0.0336 \ \Delta \qquad \text{(B66)}$$

$$m_z = 1.86 \frac{\Delta}{g} = 0.0578 \ \Delta \qquad \text{(B67)}$$

$$k^2 m_\theta = 1.61 \ k^2 \frac{\Delta}{g} = 0.050 \ k^2 \Delta \qquad \text{(B68)}$$

where Δ is displacement in lb and g is acceleration, 32.2 ft/sec^2.

These factors correspond to length to beam ratios of 4 to 1 and are therefore representative. It is felt that it is not necessary to recalculate them for each proposed icebreaker for that reason plus the fact that solutions of the icebreaking equations are comparative and not, strictly speaking, absolute.

Damping Coefficients

It is necessary to use a convenient approximation for damping coefficients in heave (h_h) and pitch (k_p). It is to be remembered that these equations for icebreaking are to be used comparatively and do not warrant the precision and complexity of some methods of determining damping coefficients.

Vosser (25) uses the following dimensionless coefficients for damping:

$$\nu_\psi^o = \frac{N_{\psi\psi}\sqrt{gL}}{\rho g L^2 \nabla} \quad \text{for pitching}$$

and

$$\nu_z^o = \frac{N_{zz}\sqrt{gL}}{\rho g \nabla} \quad \text{for heaving}$$

For pitching

$$k_p = N_{\psi\psi} = \frac{\nu_\psi^o \rho g L^2 \nabla}{\sqrt{gL}} \quad \text{lb-ft-sec}$$

$$k_p = \frac{\nu_\psi^{\,o}\ \Delta\ L^{3/2}}{g^{1/2}}$$

An average value of $\nu_\psi^{\,o}$ for relatively low pitching frequency is selected from Gerritsma's work published by Vosser (25).

Set $\nu_\psi^{\,o} = 0.10$

Then

$$k_p = \frac{0.10}{\sqrt{g}}\ \Delta\ L^{3/2}$$

$$k_p = 0.0176\ \Delta\ L^{3/2} \qquad \text{lb-ft-sec} \qquad \text{(B69)}$$

For heaving

$$k_h = N_{zz} = \frac{\nu_z^{\,o}\ \rho\ g\ \nabla}{\sqrt{g\ L}} \quad \frac{\text{lb-sec}}{\text{ft}}$$

$$k_h = \frac{\nu_z^{\,o}}{\sqrt{g}} \quad \frac{\Delta}{L^{1/2}}$$

An average value of $\nu_z^{\,o}$ based on Gerritsma's work is selected. (25)

Set $\nu_z^{\,o} = 3.0$

Then

$$k_h = \frac{3.0}{\sqrt{g}} \quad \frac{\Delta}{L^{1/2}}$$

$$k_h = 0.529\ \frac{\Delta}{L^{1/2}} \quad \frac{\text{lb-sec}}{\text{ft}} \qquad \text{(B70)}$$

Radius of Gyration

It is necessary to have a suitable value for the mass moment of inertia about the center of gravity for a pitching motion (about the y-axis). A convenient way of finding this is to know (or approximate) the radius of gyration, k.

Vosser (25) indicates that the longitudinal radius of gyration of a fully loaded ship varies between 0.22L and 0.27L. (A triangular weight distribution would have k = 0.204L).

Since an icebreaker is generally short, broad, and deep, much of its weight is toward amidships. For that reason, and as an approximation, set

$$k = 0.22\ L \tag{B71}$$

Pounds per Foot Immersion

During initial design stages the area of the water plane may be known but the next step of calculating tons per foot immersion may not have been carried out. For that reason, T_f, will be expressed in the terms of water plane coefficient.

$$T_f = L \cdot B\ (\text{Water plane coefficient})$$

$$T_f = 64.2\ L\ \alpha\ B\ \text{lb/ft} \tag{B72}$$

for sea water.

Icebreakers constructed prior to 1962 have had various water plane coefficients from 0.650 to 0.761 with an average of 0.720.

Longitudinal Metacentric Height

This value may not be known during initial design stages. In that case it would be appropriate to use

$$GM_L \approx GM_L \approx L \qquad (B73)$$

Bow Forces During Crushing

Reference to equations (B34) and (B14) will lead to values for the horizontal component of the bow force and the vertical components of the bow force respectively.

$$F_{BXC} = k_1 x^2 \qquad (B39)$$

$$F_{BZC} = k_2 x^2 \qquad (B14)$$

where k_1 and k_2 reflect the influence of σ, i_B, β, and f_k.

Sliding Phase, General

The sliding phase commences once local crushing has ceased. In other words, there is no component of velocity at the bow normal to the stem.

It is important to note that point A on the ice, the point of contact with the bow, is fixed relative to the coordinate system during the sliding phase.

Unlike the crushing phase, the friction force acts only parallel to the stem since that is the only direction of relative motion. See Figure B-XX.

If $\frac{N}{2}$ represents the force normal to the plating on each side, then the friction force can be represented by

$$\frac{F_s}{2} = f_k \frac{N}{2} \qquad F_s = f_k N \tag{B74}$$

where f_k = coefficient of kinetic friction.

As may be seen in Figure B-XXI, the force normal to the stem, in the centerline plane, may be expressed as

$$P_s = N \cos \beta \tag{B 75}$$

where β = angle between normal to plating and centerline plane

As may be seen in Figures B-XXII and B-XXIII the upward force under the bow is

$$F_{BZS} = P_s \cos (i_B + \theta) - F_s \sin (i_B + \theta)$$

Figure B-XX

Forces Acting on Bow During Sliding

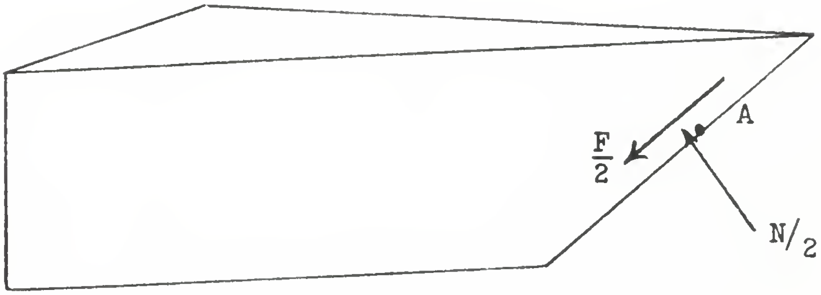


Figure B-XXI

Resolution of Friction and Normal Forces During Sliding, Looking Down Stem

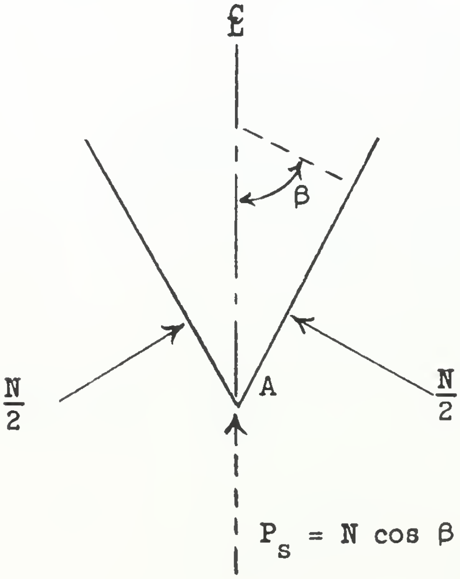
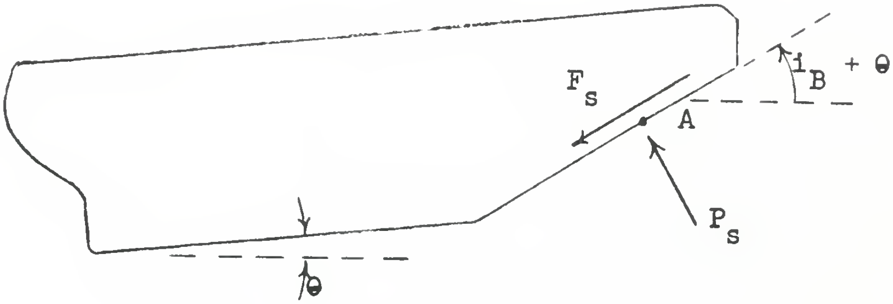


Figure B-XXII

Bow Forces During Sliding



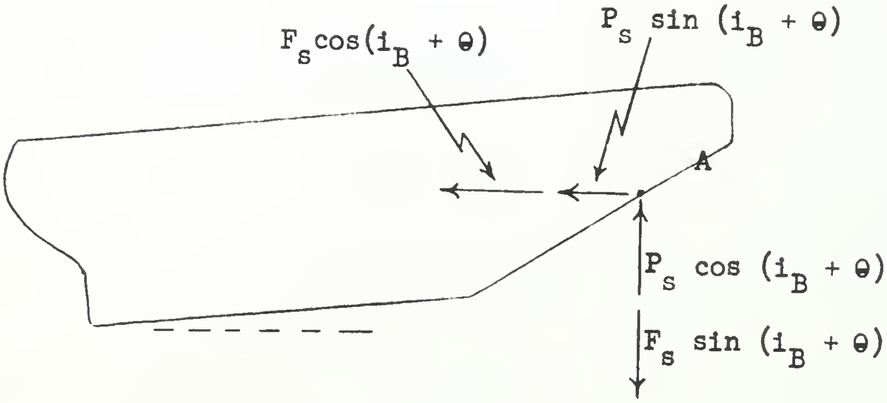
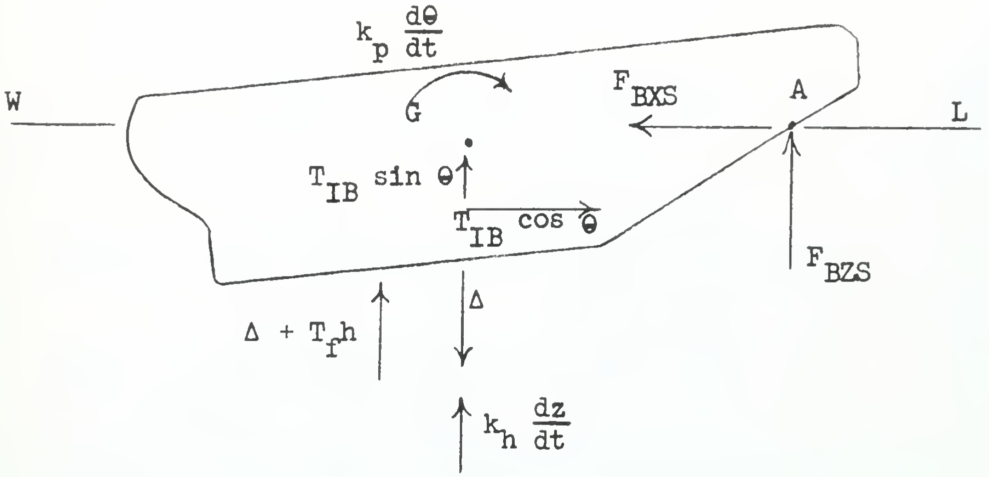


Figure B-XXIII

Free Body Diagram During Sliding Phase



$h \equiv$ increase of draft at LCF

$k_p \equiv$ coefficient of pitch damping

$k_h \equiv$ coefficient of Leave damping

$T_{IB} \equiv$ thrust available against ice

$T_f \equiv$ pounds per foot immersion

and the horizontal force to the left is

$$F_{BXS} = P_s \sin (i_B + \theta) + F_s \cos (i_B + \theta)$$

Substitution of equations (B74) and (B75) leads to

$$F_{BZS} \overset{*}{=} N \cos \beta \cos (i_B + \theta) - N f_k \sin (i_B + \theta)$$

$$F_{BZS} = N (\cos \beta \cos (i_B + \theta) - f_k \sin (i_B + \theta) \qquad \text{(B76)}$$

and

$$F_{BXS} = N \cos \beta \sin (i_B + \theta) + f_k N \cos (i_B + \theta)$$

$$F_{BXS} = N \left[\cos \beta \sin (i_B + \theta) + f_k \cos (i_B + \theta) \right] \qquad \text{(B77)}$$

For reasons indicated and justified earlier, we shall use the following approximations:

$\cos \theta = 1.0$

$\sin \theta = 0$ radians

$\tan \theta = \theta$ radians

Furthermore, the terms may also be rewritten using trigonometric substitutions.

Equation (B76) now becomes

$$F_{BZS} = N \left[\cos \beta (\cos i_B - \theta \sin i_B) - f_k (\sin i_B + \theta \cos i_B) \right]$$

$$F_{BZS} = N \left[\cos \beta \cos i_B - f_k \sin i_B) - (\cos \beta \sin i_B + f_k \cos i_B) \theta \right] \qquad \text{(B78)}$$

Equation (B77) now becomes

$$F_{BXS} = N\left[\cos\beta\,(\sin i_B + \theta)\cos i_B) + f_k\,(\cos i_B - \theta\sin i_B)\right]$$

$$F_{BXS} = N\left[(\cos\beta\sin i_B + f_k\cos i_B) + (\cos\beta\cos i_B - f_k\sin i_B)\,\theta\right] \qquad (B79)$$

Let $a_s = \cos\beta\sin i_B + f_k\cos i_B$ (B80)

and $b_s = \cos\beta\cos i_B - f_k\sin i_B$ (B81)

Then $F_{BZS} = N\,(b_s - a_s\theta)$ (B82)

and $F_{BXS} = N\,(a_s + b_s\theta)$ (B83)

$$N = \frac{F_{BZS}}{(b_s - a_s\theta)} = \frac{F_{BXS}}{(a_s + b_s\theta)}$$

$$F_{BXS} = F_{BZS}\left(\frac{a_s + b_s\theta}{b_s - a_s\theta}\right) \qquad (B84)$$

Now F_{BXS} may be expressed in terms of the vertical force, F_{BZS}. The equation can be expanded and then the terms containing θ to a degree higher than the first may be dropped. This linearizing is valid since θ (in radians) will be relatively small.

$$F_{BXS} = F_{BZS}\left[\left(\frac{a_s}{b_s} + \frac{a_s}{b_s}(\frac{a_s^2 + b_s^2}{a_s b_s}\right)\theta\right] \qquad (B85)$$

Moment Arms

The free body diagram of the icebreaker during the sliding phase is shown in Figure B-XXIII.

The distance $(GA)_z$, the moment arm for the line of action of F_{BXS} may be expressed as

$$(GA)_z = H - KG + z \tag{B86}$$

where H is the initial draft and KG is the height of the center of gravity above the keel. It must be remembered that the origin of the coordinate system is at the position G had immediately prior to initial contact (State (1)). (See Figure B-XXIV).

At State(2), at the termination of crushing, the horizontal distance to point A is

$$(GA)_{x2} = \left(\frac{L}{2} - LCG\right) - \left[(H-KG) + \frac{\left(\frac{L}{2} - LCG\right)}{\tan i_B}\right]\theta_2$$

$$+ \left[\frac{1}{\tan i_B}\right] z_2$$

See equation (B20).

Recall that, now that point A is fixed, any motion in the x-direction (beyond x_2) will reduce that value.

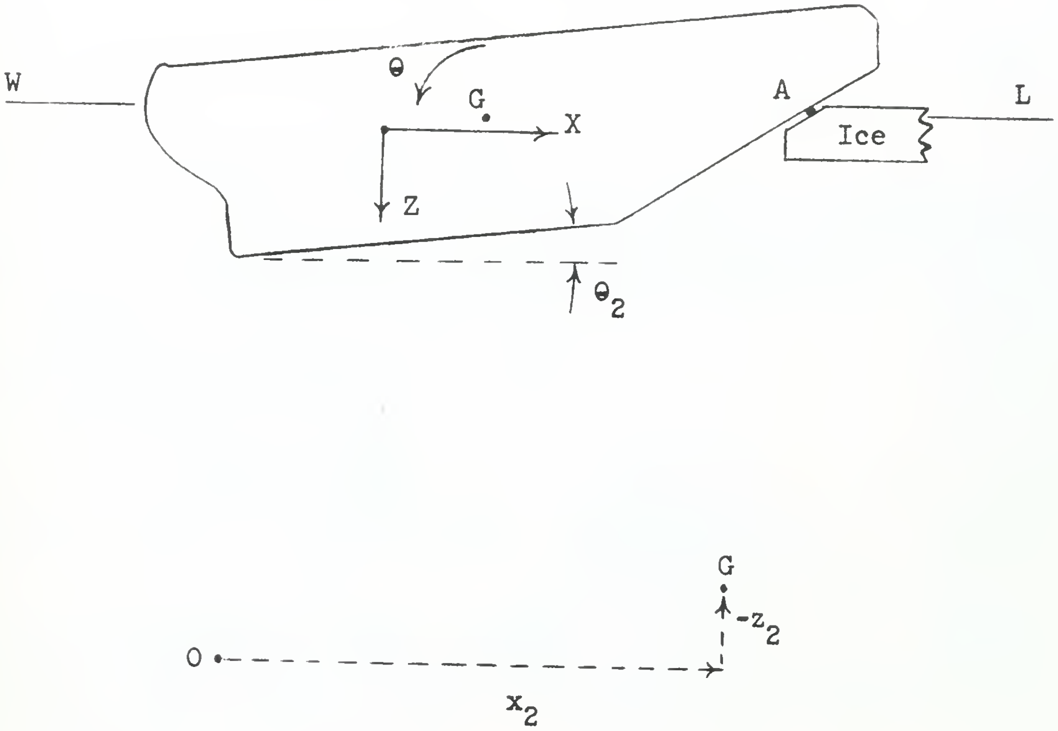
Therefore,

$$(GA)_x = \left(\frac{L}{2} - LCG\right) - (H-KG) + \frac{\left(\frac{L}{2} - LCG\right)}{\tan i_B} \quad \theta_2$$

$$+ \frac{1}{\tan i_B} z_2 - (x - x_2)$$

Figure B-XXIV

Position of State 2, the Termination of the Crushing Phase and the Commencement of the Sliding Phase



$$\text{Let } k_4 = \left(\frac{L}{2} - LCG\right) - \left[(H-KG) + \frac{(\frac{L}{2} - LCG)}{\tan i_B}\right]\theta_2$$

$$+ \frac{z_2}{\tan i_B} + x_2 \qquad \text{(B87)}$$

$$R_4 = (GA)_{x2} + x_2$$

Then $(GA)_x = k_4 - x$ (B88)

Newton's Laws of Motion During Sliding

With reference to Figure B-XXIII, Newton's Laws of motion may now be applied, for the sliding phase, rotationally about the center of gravity, in the x-direction, and in the z-direction.

In the horizontal direction

$$\sum F_x = m_x \frac{d^2x}{dt^2}$$

$$T_{IB} \cos \theta - F_{BXS} = m_x \frac{d^2x}{dt^2} \qquad \text{(B89)}$$

From equation (B28)

$$T_{IB} = T_{B)L} \left(1 - \frac{\frac{dx}{dt}}{v_1}\right)$$

From equation (B85)

$$F_{BXS} = F_{BZS} \frac{a_s}{b_s} + \frac{a_s}{b_s} \left(\frac{a_s^2 + b_s^2}{a_s b_s}\right) \theta$$

Equation (B89) may now be written as

$$T_{BOL}\,(1 - \frac{\frac{dx}{dt}}{v_1}) - F_{BZS}\left[\frac{a_s}{b_s} + \frac{a_s}{b_s}\,(\frac{a_s^2 + b_s^2}{a_s b_s})\theta\right] - m_x \frac{d^2x}{dt^2} = 0$$

$$T_{BOL} - (\frac{T_{BOL}}{v_1})\,\frac{dx}{dt} - \frac{a_s}{b_s}\,F_{BZS} - \frac{a_s}{b_s}\,(\frac{a_s^2 + b_s^2}{a_s b_s})\,F_{BZS}\theta$$

$$- m_x \frac{d^2x}{dt^2} = 0 \qquad (B90)$$

The summation of forces in the downward vertical direction (z-direction), as seen in Figure B-XXIII, may be expressed as

$$\sum F_z = m_z \frac{d^2z}{dt^2}$$

$$- F_{BZS} - T_{IB}\theta - \Delta - T_f h - k_h \frac{dz}{dt} + \Delta - m_z \frac{d^2z}{dt^2} = 0 \qquad (B91)$$

From equation (B17)

$$h = z + (LCG - LCF)\theta$$

Substitution of this and equation (B28) leads to

$$-F_{BZS} - T_{BOL}(\theta) + \frac{T_{BOL}}{v_1}\,(\frac{dx}{dt})\,\theta - T_f\,z$$

$$F_{BZS} = - T_{BOL}\theta + \frac{T_{BOL}}{v_1}\,(\frac{dx}{dt})\theta - T_f\,z$$

$$- T_f(LCG\text{-}LCF)\theta\; k_h \frac{dz}{dt} - m_z \frac{d^2z}{dt^2} \qquad (B92)$$

Substitution of equation (B92) into equation (B90 leads to

$$T_{BOL} - (\frac{T_{BOL}}{v_1})\frac{dx}{dt} + \frac{a_s}{b_s}T_{BOL}\,\theta - \frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}(\frac{dx}{dt})\,\theta + \frac{a_s}{b_s}T_f\,z$$

$$+ \frac{a_s}{b_s}\,T_f\,(LCG-LCF)\theta + k_h\frac{a_s}{b_s}\,\frac{dz}{dt} + \frac{a_s}{b_s}m_z\,\frac{d^2z}{dt^2}$$

$$+ k_5\,T_{BOL}\theta^2 - \frac{T_{BOL}}{v_1}k_5\,(\frac{dx}{dt})\,\theta^2 + k_5\,T_f\,z\,\theta + k_5\,T_f(LCG-LCF)\theta^2$$

$$+ k_5\,k_h\frac{dz}{dt}\theta + k_5 m_z\,\theta\,\frac{d^2z}{dt^2} - m_x\frac{d^2x}{dt^2} = 0 \qquad (B93)$$

where $k_5 = \frac{a_s}{b_s}\,(\frac{a_s^2 + b_s^2}{a_s b_s})$

$$k_5 = \frac{a_s^2 + b_s^2}{b_s^2} = 1 + (\frac{a_s}{b_s})^2 \qquad (B94)$$

The following products of variable terms appear in equation (B93):

$(\frac{dx}{dt})\theta$ $\qquad$ θ^2

$(\frac{dx}{dt})\theta^2$ $\qquad$ $z\,\theta$

$(\frac{dz}{dt})\,\theta$

$(\frac{d^2z}{dt^2})\,\theta$

Maclaurin's Theorem may be used to put those terms in linear form.

$$\left(\frac{dx}{dt}\right) \theta = \left(\frac{dx}{dt_2}\right) \theta_2 + \left(\frac{dx}{dt}\right)_2 \theta - \left(\frac{dx}{dt_2}\right) \theta_2 + \theta_2 \frac{dx}{dt} - \left(\frac{dx}{dt_2}\right) \theta_2$$

$$\left(\frac{dx}{dt}\right) \theta = - \left(\frac{dx}{dt_2}\right) \theta_2 + \left(\frac{dx}{dt_2}\right) \theta + \theta_2 \left(\frac{dx}{dt}\right)$$

$$\theta^2 = -\theta_2^2 + 2\, \theta_2 \theta$$

$$\left(\frac{dx}{dt}\right) \theta^2 = - \left(\frac{dx}{dt_2}\right) \theta_2 \theta + \left(\frac{dx}{dt_2}\right) \theta^2 + \theta_2 \left(\frac{dx}{dt}\right) \theta$$

$$= - \left(\frac{dx}{dt_2}\right) \theta_2\, \theta - \left(\frac{dx}{dt_2}\right) \theta_2^2 + 2 \left(\frac{dx}{dt_2}\right) \theta_2 \theta$$

$$- \left(\frac{dx}{dt_2}\right) \theta_2^2 + \left(\frac{dx}{dt_2}\right) \theta_2 \theta + \theta_2^2 \left(\frac{dx}{dt}\right)$$

$$\left(\frac{dx}{dt}\right) \theta^2 = - 2 \left(\frac{dx}{dt_2}\right) \theta_2^2 + \left[2 \left(\frac{dx}{dt_2}\right) \theta_2\right] \theta + \left[\theta_2^2\right] \left(\frac{dx}{dt}\right)$$

$$z\, \theta = - z_2 \theta_2 + z_2\, \theta + \theta_2\, z$$

$$\frac{dz}{dt} \theta = - \left(\frac{dz}{dt_2}\right) \theta_2 + \left(\frac{dz}{dt_2}\right) \theta + \theta_2 \left(\frac{dz}{dt}\right)$$

$$\left(\frac{d^2z}{dt^2}\right) \theta = - \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 + \left(\frac{d^2z}{dt^2}\right)_2 \theta + \theta_2 \left(\frac{d^2z}{dt^2}\right)$$

Equation (B93) may now be written in linear form.

$$T_{BOL} - \left(\frac{T_{BOL}}{v_1}\right) \frac{dx}{dt} + \frac{a_s}{b_s} T_{BOL} \theta + \frac{a_s}{b_s} \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt_2}\right) \theta_2 - \frac{a_s}{b_s} \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt_2}\right) \theta$$

$$- \frac{a_s}{b_s} \frac{T_{BOL}}{v_1} \theta_2 \left(\frac{dx}{dt}\right) + \frac{a_s}{b_s} T_f\, z$$

$$+\frac{a_s}{b_s} T_f \,(LCG-LCF)\theta + k_h \frac{a_s}{b_s} \frac{dz}{dt} + \frac{a_s}{b_s} m_z \frac{d^2z}{dt^2} - k_5\, T_{BOL}\, \theta_2^2 + 2\, k_5\, T_{BOL}\, \theta_2\, \theta$$

$$+\frac{2\, T_{BOL}\, k_5}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2^2 - \frac{2\, T_{BOL}\, k_5}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2\, \theta - \frac{T_{BOL}\, k_5}{v_1} \theta_2^2 \frac{dx}{dt} - k_5 T_f\, z_2 \theta_2$$

$$+ k_5\, T_f \theta_2\, z$$

$$-k_5\, T_f (LCG-LCF)\theta_2^2 + 2\, k_5 T_f (LCG-LCF)\theta_2\, \theta - k_5 k_h \left(\frac{dz}{dt}\right)_2 \theta + k_5 k_h \left(\frac{dz}{dt}\right)_2 \theta + k_5 k_h \theta_2 \frac{dz}{dt}$$

$$-k_5 m_z \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 + k_5 m_z \left(\frac{d^2z}{dt^2}\right)_2 \theta + k_5\, m_z\, \theta_2 \left(\frac{d^2z}{dt^2}\right) - m_x \frac{d^2x}{dt^2} = 0$$

$$a_{11}\, \ddot{x} + b_{11}\, \dot{x} + c_{11} x + a_{12}\, \ddot{z} + b_{12}\, \dot{z} + c_{12} z + a_{13}\, \ddot{\theta} + b_{13}\, \dot{\theta} + c_{13} \theta = d_1 \qquad \text{(B95)}$$

where

$$a_{11} = - m_x$$

$$b_{11} = \frac{-T_{BOL}}{v_1} \quad \frac{-a_s}{b_s} \quad \frac{T_{BOL}}{v_1} \theta_2 \quad \frac{-T_{BOL}\, k_5}{v_1} \quad \theta_2^2$$

$$c_{11} = 0$$

$$a_{12} = + \frac{a_s}{b_s}\, m_z + k_5\, m_z \theta_2$$

$$b_{12} = + k_h \frac{a_s}{b_s} + k_5 k_h\, \theta_2$$

$$c_{12} = \frac{+a_s}{b_s} T_f + k_5 T_f \Theta_2$$

$$a_{13} = 0$$

$$b_{13} = 0$$

$$c_{13} = \frac{+a_s}{b_s} T_{BOL} \quad \frac{-a_s}{b_s} \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 + \frac{a_s}{b_s} T_f \,(LCG\text{-}LCF)$$

$$+ 2 k_5 T_{BOL} \Theta_2 - 2 \frac{T_{BOL}}{v_1} k_5 \left(\frac{dx}{dt}\right)_2 \Theta_2 + k_5 T_f z_2$$

$$+ 2 k_5 T_f \,(LCG\text{-}LCF) \Theta_2 + k_5 k_h \left(\frac{dz}{dt}\right)_2 + k_5 m_z \left(\frac{d^2 z}{dt^2}\right)_2$$

$$d_1 = -T_{BOL} - \frac{a_s}{b_s} \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2 + k_5 T_{BOL} \Theta_2^2 - \frac{2 T_{BOL} k_5}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2^2$$

$$+ k_5 T_f z_2 \Theta_2 + k_5 T_f \,(LCG\text{-}LCF) \Theta_2^2 + k_5 k_h \left(\frac{dz}{dt}\right)_2 \Theta_2$$

$$+ k_5 m_z \left(\frac{d^2 z}{dt^2}\right)_2 \Theta_2$$

The summation of moments (counter clockwise) may be taken about the center of gravity. See Figure B-XXIII.

$$\sum M \nwarrow \; = I \frac{d^2 \Theta}{dt^2}$$

$$F_{BZS}\,(GA)_x + F_{BXS}\,(GA)_z + T_{IB}\cos\theta\,(KG-d)$$

$$-\,(\Delta + T_f h)\,GM_L\,\theta - k_p\,\frac{d\theta}{dt} - I\,\frac{d^2\theta}{dt^2} = 0 \qquad (B96)$$

From equation (B92)

$$F_{BZS} = -\,T_{BOL}\,\theta + \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta - T_f\,z - T_f\,(LCG-LCF)\theta$$

$$-k_h\,\frac{dz}{dt} - m_z\,\frac{d^2z}{dt^2}$$

From equations (B85) and (B94)

$$F_{BXS} = F_{BZS}\left[\frac{a_s}{b_s} + k_5\,\theta\right]$$

$$F_{BXS} = \frac{-a_s}{b_s}\,T_{BOL}\theta + \frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta - \frac{a_s}{b_s}\,T_f z \;\; \frac{-a_s}{b_s}\,T_f\,(LCG-LCF)\theta$$

$$-\,\frac{a_s}{b_s}\,k_h\,\frac{dz}{dt} - \frac{a_s}{b_s}\,m_z\,\frac{d^2z}{dt^2} - k_5\,T_{BOL}\,\theta^2 + k_5\,\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta^2$$

$$-\,k_5\,T_f z\,\theta - k_5\,T_f\,(LCG-LCF)\theta^2 - k_5 k_h\,\frac{dz}{dt}\,\theta - k_5 m_z\left(\frac{d^2z}{dt^2}\right)\theta$$

From equation (B28)

$$T_{IB} = T_{BOL} - \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)$$

From Equation (B17)

$$h = z + (LCG-LCF)\,\theta$$

From equation (B88)

$$(GA)_x = k_4 - x$$

$$\text{where } k_4 = \left(\frac{L}{2} - LCG\right) - \left[(H-KG) \quad \frac{+\left(\frac{L}{2} - LCG\right)}{\tan i_B}\right]\theta_2$$

$$\frac{+\ z_2}{\tan i_B} + x_2$$

From equation (B86)

$$(GA)_z = (H-KG) + z$$

These equations must now be substituted into equation (B96).

$$-k_4 T_{BOL}\,\theta + k_4 \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta - k_4 T_f\, z - k_4 T_f (LCG-LCF)\theta - k_4 k_h \frac{dz}{dt} - k_4 m_z \frac{d^2z}{dt^2}$$

$$+ T_{BOL}\,\theta_x \quad \frac{-T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta_x + T_f\, z\, x + T_f (LCG-LCF)\theta_x + k_h \frac{dz}{dt}\, x + m_z \frac{d^2z}{dt^2}\, x$$

$$\frac{-\ a_s}{b_s} T_{BOL}\,(H-KG)\theta + \frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}(H-KG)\frac{dx}{dt}\theta - \frac{-a_s}{b_s}\,T_f\,(H-KG)z - \frac{a_s}{b_s} T_f\,(LCG-LCF)(H-KG)\theta$$

$$-\frac{a_s}{b_s}\,k_h\,(H-KG)\,\frac{dz}{dt} - \frac{a_s}{b_s} m_z\,(H-KG)\,\frac{d^2z}{dt^2} - k_5\,T_{BOL}(H-KG)\theta^2 + k_5\,\frac{T_{BOL}}{v_1}(H-KG)\left(\frac{dx}{dt}\right)\theta^2$$

$$- k_5\, T_f(H\text{-}KG)z\theta - k_5 T_f\,(LCG\text{-}LCF)(H\text{-}KG)\theta^2 - k_5 k_h (H\text{-}KG)\frac{dz}{dt}\,\theta - k_5 m_z\left(\frac{d^2z}{dt^2}\right)\theta(H\text{-}KG)$$

$$\frac{-a_s}{b_s} T_{BOL}\,\theta z + \frac{a_s}{b_s}\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta z - \frac{a_s}{b_s} T_f\, z^2 - \frac{a_s}{b_s} T_f\,(LCG\text{-}LCF)\theta z - \frac{a_s}{b_s} k_h \frac{dz}{dt}\, z$$

$$- \frac{a_s}{b_s}\, m_z \frac{d^2z}{dt^2}\, z$$

$$-k_5\, T_{BOL}\,\theta^2 z + k_5 \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta^2 z - k_5 T_f \theta\, z^2 - k_5 T_f(LCG\text{-}LCF)\theta^2 z - k_5 k_h \frac{dz}{dt}\,\theta z$$

$$- k_5 m_z \left(\frac{d^2z}{dt^2}\right)\theta z$$

$$+ T_{BOL}\,(KG\text{-}d) - \frac{T_{BOL}}{v_1}(KG\text{-}d)\,\frac{dx}{dt} - \Delta GM_L\theta - T_f\, GM_L\theta z - T_f GM_L(LCG\text{-}LCF)\theta^2 - k_p \frac{d\theta}{dt}$$

$$- I\,\frac{d^2\theta}{dt^2} = 0 \qquad \text{(B98)}$$

Non-linear terms of equation (B98) must be put into linear form, as was done earlier.

$$\left(\frac{dx}{dt}\right)\theta = -\left(\frac{dx}{dt}\right)_2\theta_2 + \left(\frac{dx}{dt}\right)_2\theta + \theta_2\left(\frac{dx}{dt}\right)$$

$$\theta\, x = -\theta_2 x_2 + \theta_2\, x + x_2\,\theta$$

$$z\, s = -z_2 x_2 + z_2 s + x_2\, z$$

$$\theta^2 = -\theta_2^2 + 2\,\theta_2\,\theta$$

$$z\,\theta = -\,z_2\theta_2 + z_2\theta + \theta_2\,z$$

$$\left(\frac{dz}{dt}\right)\theta = -\left(\frac{dz}{dt}\right)_2\theta_2 + \left(\frac{dz}{dt}\right)_2\theta + \theta_2\left(\frac{dz}{dt}\right)$$

$$\left(\frac{d^2z}{dt^2}\right)\theta = -\left(\frac{d^2z}{dt^2}\right)_2\theta_2 + \left(\frac{d^2z}{dt^2}\right)\theta + \theta_2\left(\frac{d^2z}{dt^2}\right)$$

$$z^2 = -\,z_2^2 + 2\,z_2\,z$$

$$\left(\frac{d}{dt}\right)z = -\left(\frac{dz}{dt}\right)_2 z_2 + \left(\frac{dz}{dt_2}\right)z + z_2\left(\frac{dz}{dt}\right)$$

$$\left(\frac{d^2z}{dt^2}\right)z = -\left(\frac{d^2z}{dt^2}\right)_2 z_2 + \left(\frac{d^2z}{dt^2}\right)_2 z + z_2\left(\frac{d^2z}{dt^2}\right)$$

$$\left(\frac{dx}{dt}\right)\theta\,x = -\left(\frac{dx}{dt}\right)_2\theta_2\,x + \left(\frac{dx}{dt}\right)_2\theta\,x + \theta_2\left(\frac{dx}{dt}\right)x$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)\theta\,x = &-\left(\frac{dx}{dt}\right)_2\theta_2\,x - \left(\frac{dx}{dt_2}\right)\theta_2\,x_2 + \left(\frac{dx}{dt}\right)_2\theta_2\,x\\ &+\left(\frac{dx}{dt}\right)_2 x_2\,\theta - \theta_2\left(\frac{dx}{dt}\right)_2 x_2 + \theta_2\left(\frac{dx}{dt}\right)_2 x\\ &+\theta_2\,x_2\left(\frac{dx}{dt}\right)\end{aligned}$$

$$\left(\frac{dx}{dt}\right) \theta x = -2 \left(\frac{dx}{dt}\right)_2 \theta_2 x_2 + \left(\frac{dx}{dt}\right)_2 x_2 \theta + \left(\frac{dx}{dt}\right)_2 \theta_2 x$$

$$+ \theta_2 x_2 \left(\frac{dx}{dt}\right)$$

$$\left(\frac{dx}{dt}\right) \theta^2 = -2 \left(\frac{dx}{dt}\right)_2 \theta_2^2 + 2 \left(\frac{dx}{dt}\right)_2 \theta_2 \theta + \theta_2^2 \left(\frac{dx}{dt}\right)$$

$$\left(\frac{dx}{dt}\right) \theta z = -2 \left(\frac{dx}{dt}\right)_2 \theta_2 z_2 + \left(\frac{dx}{dt}\right)_2 z_2 \theta + \left(\frac{dx}{dt}\right)_2 \theta_2 z$$

$$+ \theta_2 z_2 \left(\frac{dx}{dt}\right)$$

$$\theta^2 z = -2 z_2 \theta_2^2 + 2 z_2 \theta_2 \theta + \theta_2^2 z$$

$$\theta z^2 = -2 \theta_2 z_2^2 + 2 z_2 \theta_2 z + z_2^2 \theta$$

$$\left(\frac{dz}{dt}\right) \theta z = -2 \left(\frac{dz}{dt}\right)_2 \theta_2 z_2 + \left(\frac{dz}{dt}\right)_2 z_2 \theta + \left(\frac{dz}{dt}\right)_2 \theta_2 z$$

$$+ \theta_2 z_2 \left(\frac{dz}{dt}\right)$$

$$\left(\frac{d^2z}{dt^2}\right) \theta z = -2 \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 z_2 + \left(\frac{d^2z}{dt^2}\right)_2 z_2 \theta + \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 z$$

$$+ \theta_2 z_2 \left(\frac{d^2z}{dt^2}\right)$$

$$(\frac{dx}{dt})\ \theta^2 z = -2\ (\frac{dx}{dt})_2\ \theta_2 z_2\ \theta + (\frac{dx}{dt})_2\ z_2\ \theta^2 + (\frac{dx}{dt})_2\ \theta_2\ z\ \theta$$

$$+ \theta_2 z_2\ (\frac{dx}{dt})\ \theta$$

$$(\frac{dx}{dt})\ \theta^2\ z = -2\ (\frac{dx}{dt})_2\ \theta_2 z_2\ \theta - (\frac{dx}{dt})_2\ z_2\ \theta_2^2 + 2\ (\frac{dx}{dt})_2\ z_2\ \theta_2\ \theta$$

$$- (\frac{dx}{dt})_2\ \theta_2^2\ z_2 + (\frac{dx}{dt})_2\ \theta_2\ z_2\ \theta + (\frac{dx}{dt})_2\ \theta_2^2\ z$$

$$- \theta_2\ z_2\ (\frac{dx}{dt})_2\ \theta_2 + \theta_2 z_2\ (\frac{dx}{dt})_2\ \theta$$

$$+ \theta_2 z_2 \theta_2\ (\frac{dx}{dt})$$

$$(\frac{dx}{dt})\ \theta^2\ z = -3\ (\frac{dx}{dt})_2\ z_2\ \theta_2^2 + 2\ (\frac{dx}{dt})_2\ \theta_2 z_2\ \theta + (\frac{dx}{dt})_2\ \theta_2^2\ z$$

$$+ z_2\ \theta_2^2\ (\frac{dx}{dt})$$

Equation (B98) may now be written in linear form.

$$-k_4 T_{BOL}\theta - k_4\ \frac{T_{BOL}}{v_1}\ (\frac{dx}{dt})_2 \theta_2 + k_4 \frac{T_{BOL}}{v_1}\ (\frac{dx}{dt})_2\ \theta + k_4\ \frac{T_{BOL}}{v_1} \theta_2\ (\frac{dx}{dt}) - k_4 T_f\ z$$

$$- k_4\ T_f\ (LCG - LCF)\ \theta$$

$$-k_4\, k_h \left(\frac{dz}{dt}\right) - k_4\, m_z \left(\frac{d^2z}{dt^2}\right) - T_{BOL}\, \theta_2 x_2 + T_{BOL}\, \theta_2\, x + T_{BOL}\, x_2\, \theta \;\; \frac{+2T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2\, x_2$$

$$\frac{-T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 x_2\, \theta$$

$$\frac{-T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2\, x \;\; \frac{-T_{BOL}}{v_1}\, \theta_2 x_2 \left(\frac{dx}{dt}\right) - T_f\, z_2 x_2 + T_f z_2 x + T_f\, x_2 z - T_f\, (LCG-LCF)\theta_2 x_2$$

$$+ T_f\, (LCG-LCF)\theta_2 x + T_f\, (LCG-LCF)x_2\, \theta - k_h \left(\frac{dz}{dt}\right)_2 x_2 + k_h \left(\frac{dz}{dt}\right)_2 x + k_h x_2 \left(\frac{dz}{dt}\right)$$

$$- m_z \left(\frac{d^2z}{dt^2}\right)_2 x_2 + m_z \left(\frac{d^2z}{dt^2}\right)_2 x + m_z x_2 \left(\frac{d^2z}{dt^2}\right) - \frac{a_s}{b_s} T_{BOL}\, (H-KG)\theta - \frac{a_s}{b_s} \frac{T_{BOL}}{v_1}\, (H-KG)\left(\frac{dx}{dt}\right)_2 \theta_2$$

$$+ \frac{a_s}{b_s}\, \frac{T_{BOL}}{v_1}\, (H-KG)\left(\frac{dx}{dt}\right)_2 \theta + \frac{a_s}{b_s}\, \frac{T_{BOL}}{v_1} (H-KG)\theta_2 \left(\frac{dx}{dt}\right) - \frac{a_s}{b_s} T_f (H-KG)\, z$$

$$- \frac{a_s}{b_s}\, T_f\, (LCG - LCF)(H-KG)\, \theta$$

$$- \frac{a_s}{b_s} k_h\, (H-KG)\, \frac{dz}{dt} - \frac{a_s}{b_s} m_z\, (H-KG)\, \frac{d^2z}{dt^2} + k_5\, T_{BOL}\, (H-KG)\theta_2^2 - 2\, k_5\, T_{BOL} (H-KG)\theta_2\, \theta$$

$$-2\, k_5\, \frac{T_{BOL}}{v_1} (H-KG)\left(\frac{dx}{dt}\right)_2 \theta_2^2 + 2\, k_5\, \frac{T_{BOL}}{v_1}\, (H-KG)\left(\frac{dx}{dt}\right)_2 \theta_2 \theta + k_5\, \frac{T_{BOL}}{v_1} (H-KG)\theta_2^2 \left(\frac{dx}{dt}\right)$$

$$+ k_5\, T_f\, (H-KG)\, z_2\, \theta_2$$

$$-k_5 T_f\,(H-KG)\,z_2\theta - k_5 T_f\,(H-KG)\theta_2 z + k_5 T_f (LCG-LCF)(H-KG)\theta_2^2 - 2k_5 T_f (LCG-LCF)(H-KG)\theta_2\theta$$

$$+k_5 k_h (H-KG)\left(\frac{dz}{dt}\right)_2\theta_2 - k_5 k_h (H-KG)\left(\frac{dz}{dt}\right)_2\theta - k_5 k_h (H-KG)\theta_2\left(\frac{dz}{dt}\right) + k_5\, m_z (H-KG)\left(\frac{d^2z}{dt^2}\right)_2\theta_2$$

$$-k_5 m_z (H-KG)\left(\frac{d^2z}{dt^2}\right)_2\theta - k_5 m_z (H-KG)\theta_2\left(\frac{d^2z}{dt^2}\right) + \frac{a_s}{b_s} T_{BOL}\,\theta_2 z_2 - \frac{a_s}{b_s} T_{BOL}\,\theta_2\,z - \frac{a_s}{b_s} T_{BOL}\,z_2\,\theta$$

$$-\frac{2\,a_s}{b_s}\,\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2\theta_2\,z_2 + \frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2 z_2\,\theta + \frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2\theta_2\,z$$

$$+\frac{a_s}{b_s}\,\frac{T_{BOL}}{v_1}\,\theta_2\,z_2\left(\frac{dx}{dt}\right)$$

$$+\frac{a_s}{b_s} T_f\,z_2^2 - \frac{2\,a_s}{b_s} T_f\,z_2\,z + \frac{a_s}{b_s} T_f\,(LCG-LCF)\,\theta_2\,z_2\ \frac{-a_s}{b_s} T_f\,(LCG-LCF)\theta_2\,z$$

$$-\frac{a_s}{b_s}\,T_f\,(LCG-LCF)\,z_2\,\theta$$

$$+\frac{a_s}{b_s}\,k_h\left(\frac{dz}{dt}\right)_2 z_2 - \frac{a_s}{b_s}\,k_h\left(\frac{dz}{dt}\right)_2 z - \frac{a_s}{b_s} k_h\,z_2\left(\frac{dz}{dt}\right) + \frac{a_s}{b_s}\,m_z\left(\frac{d^2z}{dt^2}\right)_2 z_2$$

$$-\frac{a_s}{b_s}\,m_z\left(\frac{d^2z}{dt^2}\right)_2 z$$

$$-\frac{a_s}{b_s} m_z z_2 \left(\frac{d^2z}{dt^2}\right) + 2 k_5 T_{BOL} \theta_2^2 z_2 - 2 k_5 T_{BOL} z_2 \theta_2 \theta - k_5 T_{BOL} \theta_2^2 z$$

$$-3 k_5^* \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 z_2 \theta_2^2$$

$$+ 2 k_5 \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2 \theta_2 z_2 \theta + k_5 \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2 \theta_2^2 z + k_5 \frac{T_{BOL}}{v_1} z_2 \theta_2^2 \left(\frac{dx}{dt}\right) + 2 k_5 T_f \theta_2 z_2^2$$

$$-2 k_5 T_f z_2 \theta_2 z - k_5 T_f z_2^2 \theta + 2 k_5 T_f (LCG-LCF) z_2 \theta_2^2 - 2 k_5 T_f (LCG-LCF)\theta_2 z_2 \theta$$

$$-k_5 T_f (LCG-LCF)\theta_2^2 z + 2 k_5 k_h \left(\frac{dz}{dt}\right)_2 \theta_2 z_2 - k_5 k_h \left(\frac{dz}{dt}\right)_2 z_2 \theta - k_5 k_h \left(\frac{dz}{dt}\right)_2 \theta_2 z$$

$$- k_5 k_h \theta_2 z_2 \left(\frac{dz}{dt}\right)$$

$$+ 2 k_5 m_z \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 z_2 - k_5 m_z \left(\frac{d^2z}{dt^2}\right)_2 \theta_2 z - k_5 m_z \left(\frac{d^2z}{dt^2}\right)_2 z_2 \theta - k_5 m_z \theta_2 z_2 \left(\frac{d^2z}{dt^2}\right) + T_{BOL} (KG-d)$$

$$-\frac{T_{BOL}}{v_1}(KG-d)\ (\frac{dx}{dt} - \Delta GM_L \theta + T_f GM_L \theta_2 z_2 - T_f GM_L z_2 \theta - T_f GM_L \theta_2 z$$

$$+ T_f GM_L (LCG-LCF)\theta_2^2$$

$$-2 T_f GM_L (LCG-LCF) \theta_2 \theta - k_p \frac{d\theta}{dt} - I \frac{d^2\theta}{dt^2} + 0 \qquad (B99)$$

$$a_{21}\ddot{x} + b_{21}\dot{x} + c_{21}x + a_{22}\ddot{z} + b_{22}\dot{z} + c_{22}z + a_2\ddot{\theta} + b_{23}\dot{\theta} + c_{23}\theta = d_2$$

(B100)

where

$$a_{21} = 0$$

$$b_{21} = + k_4 \frac{T_{BOL}}{v_1}\theta_2 - \frac{T_{BOL}}{v_1}\theta_2 x_2 + \frac{a_s}{b_s}\frac{T_{BOL}}{v_1}(H-KG)\theta_2$$

$$+ k_5 \frac{T_{BOL}}{v_1}(H-KG)\theta_2^2 + \frac{a_s}{b_s}\frac{T_{BOL}}{v_1}\theta_2 z_2 + k_5 \frac{T_{BOL}}{v_1} z_2 \theta_2^2$$

$$- \frac{T_{BOL}}{v_1}(KG-d)$$

$$c_{21} = + T_{BOL}\theta_2 \quad \frac{-T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2 \theta_2 + T_f z_2 + T_f (LCG-LCF)\theta_2$$

$$+ k_h \left(\frac{dz}{dt}\right)_2 + m_z \left(\frac{d^2z}{dt^2}\right)_2$$

$$a_{22} = - k_4 m_z + m_z x_2 - \frac{a_s}{b_s} m_z (H-KG) - k_5 m_z (H-KG)\theta_2$$

$$\frac{-a_s}{b_s} m_z z_2 - k_5 m_z \theta_2 z_2$$

$$b_{22} = -k_4 k_h + k_h x_2 \quad \frac{-a_s}{b_s} \; k_h \; (H\text{-}KG) - k_5 k_h (H\text{-}KG)\theta_2$$

$$- \frac{a_s}{b_s} \; k_h \; z_2 - k_5 k_h \; \theta_2 \; z_2$$

$$c_{22} = T_f x_2 - \frac{a_s}{b_s} T_f \; (H\text{-}KG) - k_5 \; T_f \; (H\text{-}KG)\theta_2 - k_4 T_f$$

$$- \frac{a_s}{b_s} \; T_{BOL} \; \theta_2 + \frac{a_s}{b_s} \; \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2 - \frac{2 \; a_s}{b_s} T_f \; z_2$$

$$- \frac{a_s}{b_s} \; T_f \; (LCG\text{-}LCF) \; \theta_2 \; - \frac{a_s}{b_s} \; k_h \left(\frac{dz}{dt}\right)_2 - \frac{a_s}{b_s} m_z \left(\frac{d^2 z}{dt^2}\right)_2$$

$$-k_5 \; T_{BOL} \; \theta_2^2 + k_5 \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \theta_2^2 \; - 2 \; k_5 \; T_f \; z_2 \; \theta_2$$

$$-k_5 \; T_f \; (LCG\text{-}LCF)\theta_2^2 - k_5 k_h \left(\frac{dz}{dt}\right)_2 \theta_2$$

$$-k_5 m_z \left(\frac{d^2 z}{dt^2}\right)_2 \theta_2 \; - \; T_f \; GM_L \; \theta_2$$

$$a_{23} = - I_B = - m_\theta k^2$$

$$b_{23} = - k_p$$

$$c_{23} = - k_4 \; T_{BOL} + k_4 \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 - k_4 \; T_f \; (LCG-LCF) + T_{BOL} \; x_2$$

$$- \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 x_2 + T_f \; (LCG-LCF) x_2 - \frac{a_s}{b_s} T_{BOL} \; (H-KG)$$

$$+ \frac{a_s}{b_s} \; \frac{T_{BOL}}{v_1} \; (H-KG) \left(\frac{dx}{dt}\right)_2 - \frac{a_s}{b_s} T_f \; (LCG-LCF)(H-KG)$$

$$-2 \; k_5 \; T_{BOL} \; (H-KG) \Theta_2 + 2 \; k_5 \; \frac{T_{BOL}}{v_1} \; (H-KG) \left(\frac{dx}{dt}\right)_2 \Theta_2$$

$$-k_5 \; T_f \; (H-KG) \; z_2 - 2 \; k_5 T_f \; (LCG-LCF)(H-KG) \Theta_2$$

$$-k_5 k_h \; (H-KG) \left(\frac{dz}{dt}\right)_2 - k_5 m_z (H-KG) \left(\frac{d^2 z}{dt^2}\right)_2$$

$$- \frac{a_s}{b_s} \; T_{BOL} \; z_2 + \frac{a_s}{b_s} \; \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 z_2 - \frac{a_s}{b_s} T_f \; (LCG-LCF) \; z_2$$

$$- 2 \; k_5 \; T_{BOL} \; z_2 \; \Theta_2 + 2 \; k_5 \; \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2 z_2$$

$$- k_5 T_f \; z_2^2 - 2 \; k_5 \; T_f \; (LCG-LCF) \Theta_2 \; z_2$$

$$- k_5 k_h \left(\frac{dz}{dt}\right)_2 z_2 \quad - k_5 m_z \left(\frac{d^2 z}{dt^2}\right)_2 z_2$$

$$- \Delta\, GM_L - T_f\, GM_L\, z_2 - 2\, T_f\, GM_L\, (LCG-LCF)\Theta_2$$

$$d_2 = + k_4 \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2 \quad + T_{BOL}\, \Theta_2\, x_2 - 2\, \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2 x_2$$

$$+ T_f\, z_2\, x_2 + T_f\, (LCG-LCF)\Theta_2\, x_2 + k_h \left(\frac{dz}{dt}\right)_2 x_2$$

$$+ m_z \left(\frac{d^2 z}{dt^2}\right)_2 x_2 \quad + \quad \frac{a_s}{b_s}\, \frac{T_{BOL}}{v_1}\, (H-KG)\left(\frac{dx}{dt}\right)_2 \Theta_2$$

$$- k_5\, T_{BOL}\, (H-KG)\Theta_2^2 \quad + 2\, k_5\, \frac{T_{BOL}}{v_1}\, (H-KG)\left(\frac{dx}{dt}\right)_2 \Theta_2^2$$

$$-k_5\, T_f\, (H-KG)\, z_2\, \Theta_2\, \omega - k_5\, T_f\, (LCG-LCF)(H-KG)\Theta_2^2$$

$$-k_5 k_h\, (H-KG)\left(\frac{dz}{dt}\right)_2 \Theta_2 - k_5 m_z\, (H-KG)\left(\frac{d^2 z}{dt^2}\right)_2 \Theta_2$$

$$- \frac{a_s}{b_s}\, T_{BOL}\, \Theta_2\, z_2 \quad + \quad \frac{2\, a_s}{b_s}\, \frac{T_{BOL}}{v_1} \left(\frac{dx}{dt}\right)_2 \Theta_2 z_2$$

$$-\frac{a_s}{b_s}\ T_f\ z_2^2\ -\ \frac{a_s}{b_s}\,T_f\ (LCG-LCF)\ \theta_2\ z_2$$

$$-\frac{a_s}{b_s}\ k_h\ \left(\frac{dz}{dt}\right)_2\ z_2\ -\ \frac{a_s}{b_s}\,m_z\ \left(\frac{d^2z}{dt^2}\right)_2\ z_2$$

$$-\ 2\ k_5\ T_{BOL}\ \theta_2^2\ z_2\ +\ 3\ k_5\ \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)_2\ z_2\ \theta_2^2$$

$$-2\ k_5\ T_f\ \theta_2\ z_2^2\ -\ 2\ k_5\ T_f\ (LCG-LCF)\ z_2\ \theta_2^2$$

$$-2\ k_5 k_h\ \left(\frac{dz}{dt}\right)_2\ \theta_2\ z_2\ -\ 2\ k_5 m_z\ \left(\frac{d^2z}{dt^2}\right)_2\ \theta_2\ z_2$$

$$-T_{BOL}\ (KG-d)\ -\ T_f\ GM_L\ \theta_2\ z_2\ -\ T_f\ GM_L\ (LCG-LCF)\theta_2^2$$

Location Geometry

During the sliding phase point A, on the ice, is fixed relative to our coordinate system. Since the bow maintains contact with this point there must exist a definite relationship among $(\theta - \theta_2)$, $(z - z_2)$, and $(x - x_2)$. These relationships are illustrated in Figure B-XXV.

If the ship is rotated $(\theta - \theta_2)$ counterclockwise and raised $-(z - z_2)$, the ship must be advanced $(x - x_2)$ in order to maintain contact.

It may be seen that

$$\tan (i_B + \theta) = \frac{(GA)_{x2} (\theta - \theta_2) - (z - z_2)}{(x - x_2) - (GA_z)_2 (\theta - \theta_2)}$$

$$(x - x_2) = \frac{(GA)_{x2} (\theta - \theta_2) - (z - z_2)}{\tan (i_B + \theta)} + (GA)_{z2} (\theta - \theta_2)$$

$$(x - x_2) - \frac{(GA)_{x2}(\theta - \theta_2)}{\tan (i_B + \theta)} + \frac{(z - z_2)}{\tan (i_B + \theta)} + (GA)_{z2}(\theta - \theta_2) = 0 \tag{B101}$$

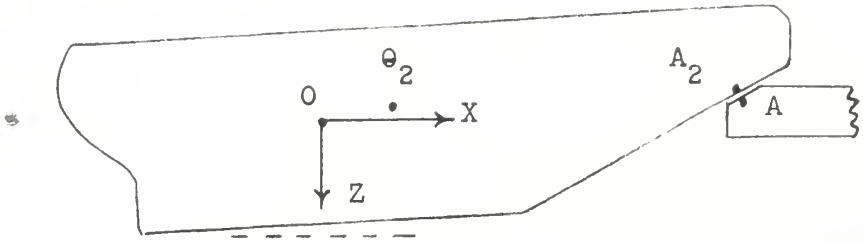
Set $(GA)_{x2} = k_6 = (\frac{L}{2} - LCG) - \left[(H\text{-}KG) + \frac{(\frac{L}{2} - LCG)}{\tan i_B} \right] \theta_2$

$$+ \frac{z_2}{\tan i_B}$$

Set $(GA)_{z2} = k_7 = (H\text{-}KG) + z_2$

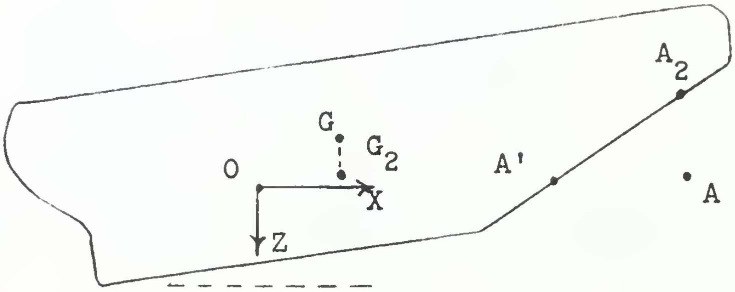
Figure B-XXV

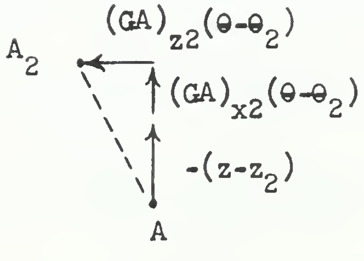
Illustration of Position Geometry

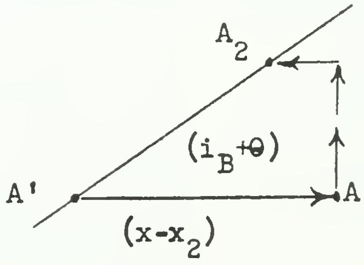


State 2

Rotate $(\theta - \theta_2)$ about the center of gravity and then raise it $-(z-z_2)$.







From equation (B19)

$$\frac{1}{\tan\,(i_B + \theta)} = \frac{1}{\tan\, i_B} - \frac{\theta}{\sin^2\, i_B}$$

Then equation (B101) becomes

$$x - x_2 - \frac{k_6\,(\theta - \theta_2}{\tan\, i_B} + \frac{k_6(\theta - \theta_2)\,\theta}{\sin^2\, i_B} + \frac{(z - z_2)}{\tan\, i_B}$$

$$\frac{-\,(z - z_2)\,\theta}{\sin^2\, i_B} + k_7\,(\theta - \theta_2) = 0$$

$$x - x_2\ \frac{-\,k_6}{\tan\, i_B}\,\theta + \frac{k_6\theta_2}{\tan\, i_B} + \frac{k_6}{\sin^2\, i_B}\,\theta^2 - \frac{k_6\,\theta_2}{\sin^2\, i_B}\,\theta$$

$$+ \frac{z}{\tan\, i_B} - \frac{z_2}{\tan\, i_B} - \frac{z\,\theta}{\sin^2\, i_B} + \frac{z_2\,\theta}{\sin^2 i_B} + k_7\,\theta - k_7\theta_2 = 0$$

As was done previously, the non-linear terms must be linearized.

$$\theta^2 = -\,\theta_2^2 + 2\,\theta_2\,\theta$$

$$z\,\theta = -\,z_2\theta_2 + z_2\,\theta + \theta_2 z$$

$$x - x_2\ \frac{-\,k_6}{\tan\, i_B}\,\theta + \frac{k_6}{\tan\, i_B}\,\theta_2 - \frac{k_6}{\sin^2 i_B}\,\theta_2^2$$

$$+ \frac{2\, k_6\, \theta_2}{\sin^2 i_B}\, \theta \quad \frac{-\, k_6\, \theta_2}{\sin^2 i_B}\, \theta + \frac{z}{\tan i_B} - \frac{z_2}{\tan i_B}$$

$$+ \frac{z_2\, \theta_2}{\sin^2 i_B} \quad \frac{z_2}{\sin^2 i_B}\, \theta - \frac{\theta_2}{\sin^2 i_B}\, z$$

$$+ \frac{z_2\, \theta}{\sin^2 i_B} + k_7\, \theta - k_7 \theta_2 = 0$$

$$x - x_2 \quad \frac{-\, k_6}{\tan i_B}\, \theta + \frac{k_6}{\tan i_B}\, \theta_2 \quad \frac{-k_6 \theta_2^2}{\sin^2 i_B} + \frac{k_6\, \theta_2}{\sin^2 i_B}\, \theta$$

$$+ \frac{z}{\tan i_B} - \frac{z_2}{\tan i_B} + \frac{z_2\, \theta_2}{\sin^2 i_B} \quad \frac{\theta_2}{\sin^2 i_B}\, z$$

$$+ k_7\, \theta - k_7\, \theta_2 = 0 \tag{B102}$$

Equation (B102) may be written as

$$a_{31}\, \ddot{x} + b_{31}\, \dot{x} + c_{31} x + a_{32}\, \ddot{z} + b_{32} \dot{z} + c_{32}\, z$$

$$+ a_{33} \ddot{\theta} + b_{33}\, \dot{\theta} + c_{33}\, \theta = d_3 \tag{B103}$$

where

$a_{31} = 0$

$b_{31} = 0$

$c_{31} = \pm 1$

$a_{32} = 0$

$b_{32} = 0$

$$c_{32} = \frac{1}{\tan i_B} - \frac{\Theta_2}{\sin^2 i_B}$$

$a_{33} = 0$

$b_{33} = 0$

$$c_{33} = \frac{-k_6}{\tan i_B} + \frac{k_6 \Theta_2}{\sin^2 i_B} + k_7$$

$$d_3 = + x_2 \quad \frac{-k_6 \Theta_2}{\tan i_B} + \frac{k_6 \Theta_2^2}{\sin^2 i_B} + \frac{z_2}{\tan i_B}$$

$$- \frac{z_2 \Theta_2}{\sin^2 i_B} + k_7 \Theta_2$$

where $k_6 = (GA)_{x2}$ $\qquad$ $k_1 = (GA)_{z2}$

[illegible]

[illegible]

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Simultaneous Equations of Sliding

Grouping the three equations,

Equation (B95)

$$a_{11}\ddot{x} + b_{11}\dot{x} + (0)x + a_{12}\ddot{z} + b_{12}\dot{z} + c_{12}z + (0)\ddot{\theta} + (0)\dot{\theta} + c_{13}\theta = d_1$$

Equation (B100)

$$(0)\ddot{x} + b_{21}\dot{x} + c_{21}x + a_{22}\ddot{z} + b_{22}\dot{z} + c_{22}z + a_{23}\ddot{\theta} + b_{23}\dot{\theta} + c_{23}\theta = d_2$$

Equation (B103)

$$(0)\ddot{x} + (0)\dot{x} + c_{31}x + (0)\ddot{z} + (0)\dot{z} + c_{32}z + (0)\ddot{\theta} + (0)\dot{\theta} + c_{33}\theta = d_3$$

Rewriting in operational form.

$$(a_{11}D^2 + b_{11}D + 0)\,x + (a_{12}D^2 + b_{12}D + c_{12})\,z + (0\,D^2 + 0\,D + c_{13})\theta = d_1$$

$$(0\,D^2 + b_{21}D + c_{21})x + (a_{22}D^2 + b_{22}D + c_{22})\,z + (a_{23}D^2 + b_{23}D + c_{23})\theta = d_2$$

$$(c_{31})x + (c_{32})z + (c_{33})\theta = d_3$$

Laplace Transforms

$f(t)$	$F(\ ) = \ [f(t)]$
$\frac{d}{dt}\ [f(\ )]$	$s\ \mathcal{L}\ [f(\ )] - f(0^+)$
$\frac{d^2}{dt^2}\ [f(\ )]$	$s^2\ \mathcal{L}\,[f(\ )] - s\ f(0^+) - \frac{df}{dt}(0^+)$

For example, if $f(t) = x$

$$\mathcal{L}\ [a\ D^2 x] = a\ s^2 \quad (x) - a\ s\ x_o - a\ \dot{x}_o$$

$$\mathcal{L}\ [\ D\ x] = b\ s\ \mathcal{L}(x) - b\ x_o$$

$$\mathcal{L}\ [c\ x] = c\ \mathcal{L}(x)$$

For further example, if $x = x_2$ and $\frac{dx}{dt} = (\frac{dx}{dt})_2$ at $t = 0$,

$$\begin{aligned}\mathcal{L}\left[a\ D^2 + b\ D + c\right] x = {} & a\ s^2\ \mathcal{L}(x) - a\ s\ x_2 - a\ \dot{x}_2 \\ & + b\ s\ \mathcal{L}(x) - b\ x_2 \\ & + c\ \mathcal{L}\ (x)\end{aligned}$$

$$\mathcal{L}\left[a\ D^2 + b\ D + c\right] x = (a\ s^2 + b\ s + c)\,\mathcal{L}\,[x] - a\ s\ x_2 - a\ \dot{x}_2 - b\ x_2$$

The three simultaneous equations (B95) (B100), and (B103) may be written as follows using LaPlace Transforms.

$$(a_{11}s^2 + b_{11}s)\, L[x] + (a_{12}s^2 + b_{12}s + c_{12})\, L[z] + (c_{13})\, L[\theta] =$$

$$a_{11}s\, x_2 + a_{11}\dot{x}_2 + b_{11}x_2 + a_{12}s\, z_2 + a_{12}\dot{z}_2 + b_{12}z_2 + \frac{d_1}{s}$$

$$(b_{21}s + c_{21})\, L[x] + (a_{22}s^2 + b_{22}s + c_{22})\, L[z] + (a_{23}s^2 + b_{23}s + c_{23})\, L[\theta] =$$

$$b_{21}x_2 + a_{22}s\, z_2 + a_{22}\dot{z}_2 + b_{22}z_2 + a_{23}s\, \theta_2 + a_{23}\dot{\theta}_2 + b_{23}\theta_2 + \frac{d_2}{s}$$

$$(c_{21})\, L[x] + (c_{32})\, L[z] + (c_{33})\, L[\theta] = \frac{d_3}{s}$$

Let $d_{11} = a_{11}\dot{x}_2 + b_{11}x_2 + a_{12}\dot{z}_2 + b_{12}z_2$

$d_{12} = a_{11}x_2 + a_{12}z_2$ (B104)

$d_{13} = d_1$

$d_{21} = b_{21}x_2 + a_{22}\dot{z}_2 + b_{22}z_2 + a_{23}\dot{\theta}_2 + b_{23}\theta_2$

$d_{22} = a_{22}z_2 + a_{23}\theta_2$

$d_{23} = d_2$

$d_{33} = d_3$

Then the simultaneous equations may be written in their shortened form.

$$(a_{11} s^2 + b_{11} s)\mathcal{L}[x] + (a_{12} s^2 + b_{12} s + c_{12})\mathcal{L}[z] + (c_{13})\mathcal{L}[\theta] = d_{11} + d_{12} s + \frac{d_{13}}{s} \tag{B105}$$

$$(b_{21} s + c_{21})\mathcal{L}[x] + (a_{22} s^2 + b_{22} s + c_{22})\mathcal{L}[z] + (a_{23} s^2 + b_{23} s + c_{23})\mathcal{L}[\theta]$$

$$= d_{21} + d_{22} s + \frac{d_{23}}{s} \tag{B106}$$

$$(c_{31})\mathcal{L}[x] + (c_{32})\mathcal{L}[z] + (c_{33})\mathcal{L}[\theta] = \frac{d_{33}}{s}$$

Using determinant form, the simultaneous Laplace equations may be solved for $\mathcal{L}[x]$, $\mathcal{L}[z]$, and $\mathcal{L}[\theta]$.

$$|x| = N_1 = \frac{\begin{array}{lll} (d_{11} + d_{12}s + \frac{d_{13}}{s}) & (a_{12}s^2 + b_{12}s + c_{12}) & (c_{13}) \\ (d_{21} + d_{22}s + \frac{d_{23}}{s}) & (a_{22}s^2 + b_{22}s + c_{22}) & (a_{23}s^2 + b_{23}s + c_{23}) \\ (\frac{d_{33}}{s}) & (c_{32}) & (c_{33}) \end{array}}{\begin{array}{lll} (a_{11}s^2 + b_{11}s) & (a_{12}s^2 + b_{12}s + c_{12}) & (c_{13}) \\ (b_{21}s + c_{21}) & (a_{22}s^2 + b_{22}s + c_{22}) & (a_{23}s^2 + b_{23}s + c_{23}) \\ (c_{31}) & (c_{32}) & (c_{33}) \end{array}}$$

$$N_1 = d_{11}a_{22}c_{33}s^2 + d_{11}b_{22}c_{33}s + d_{11}c_{22}c_{33} + d_{12}a_{22}c_{33}s^3 + d_{12}b_{22}c_{33}s^2 + d_{12}c_{22}c_{33}s$$

$$+ d_{13}a_{22}c_{33}s + d_{13}b_{22}c_{33} + d_{13}c_{22}c_{33}\,\frac{1}{2} + a_{12}a_{23}d_{33}s^3 + a_{12}b_{23}d_{33}s^2 + a_{12}c_{23}d_{33}s$$

$$+ b_{12}a_{23}d_{33}s^2 + b_{12}b_{23}d_{33}s + b_{12}c_{23}d_{33} + c_{12}a_{23}d_{33}s + c_{12}b_{23}d_{33} + c_{12}c_{23}d_{33}\,\frac{1}{s}$$

$$+ c_{13}d_{21}c_{32} + c_{13}d_{22}c_{32}s + c_{13}d_{23}c_{32}\,\frac{1}{2} - d_{11}a_{23}c_{32}s^2 - d_{11}b_{23}c_{32}s - d_{11}c_{23}c_{32}$$

$$- d_{12}a_{23}c_{32}s^3 - d_{12}b_{23}c_{32}s^2 - d_{12}c_{23}c_{32}s - d_{13}a_{23}c_{32}s - d_{13}b_{23}c_{32} - d_{13}c_{23}c_{32}\,\frac{1}{s}$$

$$-a_{12}d_{21}c_{33}s^2-a_{12}d_{22}c_{33}s^3-a_{12}d_{23}c_{33}s-b_{12}d_{21}c_{33}s-b_{12}d_{22}c_{33}s^2-b_{12}d_{23}c_{33}$$

$$-c_{12}d_{21}c_{33}-c_{12}d_{22}c_{33}s-c_{12}d_{23}c_{33}\,\frac{1}{2}-c_{13}a_{22}d_{33}s-c_{13}b_{22}d_{33}-c_{13}d_{22}d_{33}\,\frac{1}{s}$$

$$N_1 = N_{13}s^3 + N_{12}s^2 + N_{11}s + N_{10} + N_{09}\,\frac{1}{2} \qquad \text{(B108)}$$

$$N_{13} = (d_{12}a_{22}c_{33} + a_{12}a_{23}d_{33} - d_{12}a_{23}c_{32} - a_{12}d_{22}c_{33})$$

$$N_{12}=(d_{11}a_{22}c_{33}+d_{12}b_{22}c_{33}+a_{12}b_{23}d_{33}+b_{12}a_{23}d_{33}-d_{11}a_{23}c_{32}-d_{12}b_{23}c_{32}$$

$$-a_{12}d_{21}c_{33} - b_{12}d_{22}c_{33})$$

$$N_{11}=(d_{11}b_{22}c_{33}+d_{12}c_{22}c_{33}+d_{13}a_{22}c_{33}+a_{12}c_{23}d_{33}+b_{12}b_{23}d_{33}+c_{12}a_{23}d_{33})$$

$$+c_{13}d_{22}c_{32}-d_{11}b_{23}c_{32}-d_{12}c_{23}c_{32}-d_{13}a_{23}c_{32}-a_{12}d_{23}c_{33}$$

$$-b_{12}d_{21}c_{33}-c_{12}d_{22}c_{33}-c_{13}a_{22}d_{33})$$

$$N_{10}=(d_{11}c_{22}c_{33}+d_{13}b_{22}c_{33}+b_{12}c_{23}d_{33}+c_{12}b_{23}d_{33}+c_{13}d_{21}c_{32}-d_{11}c_{23}c_{32}$$

$$-d_{13}b_{23}c_{32}-b_{12}d_{23}c_{33}-c_{12}d_{21}c_{33}-c_{13}b_{22}d_{33})$$

$$N_{09}=(d_{13}c_{22}c_{33}+c_{12}c_{23}d_{33}+c_{13}d_{23}c_{32}-d_{13}c_{23}c_{32}-c_{12}d_{23}c_{33}-c_{13}c_{22}d_{33})$$

$$\text{Den.} = a_{11}a_{22}c_{33}s^4 + a_{11}b_{22}c_{33}s^3 + a_{11}c_{22}c_{33}s^2 + b_{11}a_{22}c_{33}s^3 + b_{11}b_{22}c_{33}s^2$$

$$+ b_{11}c_{22}c_{33}\, s$$

$$+ a_{12}a_{23}c_{31}s^4 + a_{12}b_{23}c_{31}s^3 + a_{12}c_{23}c_{31}s^2 + b_{12}a_{23}c_{31}s^3 + b_{12}b_{23}c_{31}s^2$$

$$+ b_{12}c_{23}c_{31}s$$

$$+ c_{12}a_{23}c_{31}s^2 + c_{12}b_{23}c_{31}s + c_{12}c_{23}c_{31} + c_{13}b_{21}c_{32}s + c_{13}c_{21}c_{32}$$

$$- a_{11}a_{23}c_{32}s^4 - a_{11}b_{23}c_{32}s^3 - a_{11}c_{23}c_{32}s^2 - b_{11}a_{23}c_{32}s^3 - b_{11}b_{23}c_{32}s^2$$

$$- b_{11}c_{23}c_{32}s$$

$$-a_{12}b_{21}c_{33}s^3 - a_{12}c_{21}c_{33}s^2 - b_{12}b_{21}c_{33}s^2 - b_{12}c_{21}c_{33}s - c_{12}b_{21}c_{33}s - c_{12}c_{21}c_{33}$$

$$-c_{13}a_{22}c_{31}s^2 - c_{13}b_{22}c_{31}s - c_{13}c_{22}c_{31}$$

$$\text{Den} = D_4s^4 + D_3\, s^3 + D_2s^2 + D_1s + D_o \tag{B109}$$

$$D_4 = (a_{11}a_{22}c_{33} + a_{12}a_{23}c_{31} - a_{11}a_{23}c_{32})$$

$$D_3 = (a_{11}b_{22}c_{33} + b_{11}a_{22}c_{33} + a_{12}b_{23}c_{31} + b_{12}a_{23}c_{31} - a_{11}b_{23}c_{32} - b_{11}a_{23}c_{32}$$

$$- a_{12}b_{21}c_{33})$$

$$D_2 = (a_{11}c_{22}c_{33} + b_{11}b_{22}c_{33} + a_{12}c_{23}c_{31} + b_{12}b_{23}c_{31} + c_{12}a_{23}c_{31} - a_{11}c_{23}c_{32}$$

$$- b_{11}b_{23}c_{32} - a_{12}c_{21}c_{33} - b_{12}b_{21}c_{33} - c_{13}a_{22}c_{31})$$

$$D_1 = (b_{11}c_{22}c_{33} + b_{12}c_{23}c_{31} + c_{12}b_{23}c_{31} + c_{13}b_{21}c_{32} - b_{11}c_{23}c_{32} - b_{12}c_{21}c_{33}$$

$$- c_{12}b_{21}c_{33} - c_{13}b_{22}c_{31})$$

$$D_0 = (c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{12}c_{21}c_{33} - c_{13}c_{22}c_{31})$$

$$\mathcal{L}[x] = \frac{N_1}{\text{Den}} = \frac{N_{13}s^3 + N_{12}\,s^2 + N_{11}s + N_{10} + N_{09}\frac{1}{s}}{D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0}$$

$$\mathcal{L}[x] = \frac{N_{13}s^4 + N_{12}s^3 + N_{11}s^2 + N_{10}\,s + N_{09}}{s\quad D_4s^4 + D_5s^3 + D_2s^2 + D_1s + D_0} \qquad \text{(B110)}$$

Solution of the LaPlace Transforms

$$\mathcal{L}[x] = \frac{\frac{N_{13}}{D_4} s^4 + \frac{N_{12}}{D_4} s^3 + \frac{N_{11}}{D_4} s^2 + \frac{N_{10}}{D_4} s + \frac{N_{09}}{D_4}}{\frac{D_4}{D_4} s^5 + \frac{D_3}{D_4} s^4 + \frac{D_2}{D_4} s^3 + \frac{D_1}{D_4} s^2 + \frac{D_0}{D_4} s} \qquad \text{(B111)}$$

Let $a_{4_x} = \frac{N_{13}}{D_4} \quad a_{3_x} = \frac{N_{12}}{D_4} \quad a_{2_x} = \frac{N_{11}}{D_4} \quad a_{1_x} = \frac{N_{10}}{D_4} \quad a_{0_x} = \frac{N_{09}}{D_4}$

(B112)

$$b_4 = \frac{D_3}{D_4} \quad b_3 = \frac{D_2}{D_4} \quad b_2 = \frac{D_1}{D_4} \quad b_1 = \frac{D_0}{D_4} \quad b_o = 0$$

Then,

$$\mathcal{L}[x] = \frac{a_{4_x} s^4 + a_{3_x} s^3 + a_{2_x} s^2 + a_{1_x} s + a_{0_x}}{s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}$$

$$\mathcal{L}[x] = \frac{a_p s^p + a_{p-1} s^{p-1} + \ldots}{s^q + b_{q-1} s^{q-1} + \ldots} = \frac{A(s)}{B(s)}$$

$$\mathcal{L}[x] = \frac{A(s)}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)} \qquad \text{(B114)}$$

Since $b_o = 0 \qquad s_1 = 0$

Biquadratic Solution

Transform

$$s^4 + b_4 s^3 + b_3 s^2 + b_2 s + b_1$$

$$\text{to} \quad \left[(s + B_1)^2 + A_1^2\right] \left[(s + B_2)^2 + A_2^2\right] \qquad \text{(B115)}$$

$$\text{or} \quad \left[(s + \alpha_3)^2 + \beta_3^2\right] \left[(s + \alpha_4)^2 + \beta_4^2\right]$$

$$(s + B_1)^2 + A_1^2 = s^2 + 2\,B_1 s + (A_1^2 + B_1^2) \qquad \text{(B116)}$$

$$\begin{array}{rl}
 & s^2 + (b_4 - 2B_1)s + \left[b_3 - (A_1^2 + B_1^2) - 2\,B_1(b_4 - 2B_1)\right] \\
s^2 + 2\,B_1 s + (A_1^2 + B^2) & \overline{\big)\, s^4 + b_4\, s^3 + b_3\, s^2 + b_2 s + b_1} \\
 & \underline{s^4 + 2\,B_1\, s^3 + (A_1^2 + B_1^2)\, s^2} \\
 & (b_4 - 2\,B_1)s^3 + \left[b_5 - (A_1^2 + B_1^2)\right] s^2 + b_2 s + b_1 \\
 & \underline{(b_4 - 2\,B_1)s^3 + 2B_1(b_4 - 2B_1)s^2 + (b_4 - 2B_1)(A_1^2 + B_1^2)\, s} \\
 & \left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right] s^2 + \left[b_2 - (b_4 - 2B_1)(A_1^2 + B_1^2)\right] s + b_1 \\
 & \\
 & \left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right] s^2 + \left[2B_1\right]\left[b_3 - (A_1^2 + B_1^2) - 2B_1(b_4 - 2B_1)\right] s \\
 & \qquad + \left[b_3 - \dots\right]\left[A_1^2 + B_1^2\right]
\end{array}$$

$$2\,B_2 = (b_4 - 2B_1)$$

$$(A_2^2 + B_2^2) = b_3 - (A_1^2 + B_1^2) - 2\,B_1(b_4 - 2B_1)$$

$$A_2 = \sqrt{b_3 - (A_1^2 + B_1^2) - 2\,B_1\,(b_4 - 2B_1) - B_2^2} \qquad \text{(B117)}$$

The remainder must equal zero.

Therefore,

$$\left[b_2 - (b_4 - 2B_1)(A_1^2 + B_1^2)\right] - \left[b_3 - (A_1^2 + B_1^2) - 2B_1 (b_4 - 2B_1)\right]\left[2B_1\right] = 0 \qquad \text{(B118)}$$

$B_1 \neq 0$ B_1 must be pos. to get $e^{-B_1 t}$, B_2 must be pos. too so $(b_4 - 2B_1) > 0$

or $\frac{b_4}{2} > B_1$

So $0 < B_1 < \frac{b_4}{2}$

$$b_2 - b_4 A_1^2 + 2A_1^2 B_1 - b_4 B_1^2 + 2B_1^3 - 2b_3 B_1 + 2A_1^2 B_1 + 2B_1^3 + 4b_4 B_1^2 - 8B_1^3 = 0$$

$$b_2 - b_4 A_1^2 + 4A_1^2 B_1 + 3b_4 B_1^2 - 4B_1^3 - 2b_3 B_1 = 0 \qquad \text{(B119)}$$

$$b_1 - \left[b_3 - (A_1^2 + B_1^2) - 2B_1 (b_4 - 2B_1)\right]\left[A_1^2 + B_1^2\right] = 0$$

$$b_1 - \left[b_3 - A_1^2 - B_1^2 - 2b_4 B_1 + 4B_1^2\right]\left[A_1^2 + B_1^2\right] = 0$$

$$b_1 - b_3 A_1^2 + A_1^4 + B_1^2 A_1^2 + 2b_4 A_1^2 B_1 - 4A_1^2 B_1^2 - b_3 B_1^2 + A_1^2 B_1^2 + B_1^4 + 2b_4 B_1^3 - 4B_1^4 = 0$$

(B120)

$$b_1 - b_3 A_1^2 + A_1^4 - 2A_1^2 B_1^2 + 2b_4 A_1^2 B_1 - b_3 B_1^2 - 3B_1^4 + 2b_4 B_1^3 = 0$$

From (E119), $A_1^2\ (4B_1 - b_4) = -b_2 - 3b_4B_1^2 + 4B_1^3 + 2b_3B_1$

$$A_1^2 = \frac{-b_2-3b_4B_1^2 + 4B_1^3 + 2b_3B_1}{(4B_1 - b_4)} \text{(Note that } B_1 \neq \frac{b_4}{4} \text{ or } A_1 \quad \text{oo)}$$

$$A_1^4 = \frac{1}{(16B_1^2 - 8b_4B_1 + b_4^2)}\ (+b_2^2 + 3b_2b_4B_1^2 - 4b_2B_1^3 - 2b_2b_3B_1$$

$$+3b_2b_4B_1^2 \qquad +9b_4^2B_1^4 - 12b_4B_1^5 - 6b_3b_4\ B_1^3$$

$$+8b_3B_1^4 \qquad -4b_2B_1^3 \qquad -12b_4B_1^5 \qquad +16B_1^6$$

$$+4b_3^2\ B_1^2 \qquad +8b_3B_1^4 \qquad -2b_2b_3B_1 \qquad -6b_3b_4B_1^3$$

$$A_1^4 = \frac{(+b_2^2+9b_4^2B_1^4+16B_1^6+4b_3^2B_1^2+6b_2b_4B_1^2\ -8b_2B_1^3-4b_2b_3B_1+16b_3B_1^4-24b_4B_1^5-12b_3b_4B_1^3)}{(16B_1^2-8b_4B_1+b_4^2)}$$

$$b_1 + A_1^2(-b_3-2B_1^2 + 2b_4B_1) + A_1^4 - b_3B_1^2 - 3B_1^4 + 2b_4B_1^3 = 0$$

$$b_1(16B_1^2-8b_4B_1+b_4^2) + (4B_1-b_4)(-b_2-3b_4B_1^2+4B_1^3 + 2b_3B_1)(-b_3-2B_1^2+2b_4B_1)$$

$$+ (-b_3B_1^2 - 3B_1^4 + 2b_4B_1^3)(16B_1^2-8b_4B_1+b_4^2)$$

$$+b_2^2+9b_4^2B_1^4 + 16B_1^6+4b_3^2B_1^2 + 6b_2b_4B_1^2-8b_2B_1^3-4b_2b_3B_1+16b_3B_1^4-24b_4B_1^5$$

$$-12\ b_3b_4B_1 = 0$$

$$16\, b_1B_1^2 - 8b_1b_4B_1 + b_1b_4^2 + (4B_1-b_4)(+b_2b_3 + 3b_3b_4B_1^2 - 4b_3B_1^3 - 2b_3^2B_1 + 2b_2B_1 2$$

$$+ 6\, b_4B_1^4 - 8B_1^5 - 4\, b_3B_1^3$$

$$-2b_2b_4B_1 - 6b_4^2B_1^3 + 8b_4B_1^4 + 4b_3b_4B_1^2) - 16b_3B_1^4 + 8b_3b_4B_1^3 - b_3b_4^2B_1^2 - 48B_1^6$$

$$+ 24\, b_4B_1^5 - 3b_4^2\, B_1^4$$

$$+ 32\, b_4B_1^5 - 16\, b_4^2\, B_1^4 + 2b_4^3B_1^3 + b_2^2 + 9b_4^2\, B_1^4 + 16B_1^6 + 4b_3^2B_1^2 + 6b_2b_4B_1^2$$

$$-8\, b_2B_1^3 - 4b_2b_3B_1 + 16\, b_3B_1^4$$

$$- 24\, b_4B_1^5 - 12\, b_3b_4\, B_1^3 = 0$$

$$16\, b_1B_1^2 - 8b_1b_4B_1 + b_1b_4^2 + 4\, b_2b_3B_1 + 12b_3b_4B_1^3 - 16b_3B_1^4 - 8b_3^2\, B_1^2 + 8b_2B_1^3$$

$$+ 24b_4B_1^5 - 32\, B_1^6 - 16\, b_3B_1^4$$

$$-8\, b_2b_4B_1^2 - 24\, b_4^2B_1^4 + 32b_4B_1^5 + 16b_3b_4B_1^3 - b_2b_3b_4 - 3b_3b_4^2B_1^2 + 4b_3b_4B_1^3$$

$$+ 2b_3^2\, b_4B_1 - 2b_2b_4B_1^2 - 6b_4^2\, B_1^4$$

$$+8\, b_4B_1^5 + 4b_3b_4B_1^3 + 2b_2b_4^2B_1 + 6b_4^3\, B_1^3 - 8b_4^2\, B_1^4 - 4b_3b_4^2\, B_1^2 - 16b_3B_1^4$$

$$+ 8\, b_3b_4B_1^3 - b_3b_4^2\, B_1^2$$

$$- 48B_1^6 + 24b_4B_1^5 - 3b_4^2B_1^4 + 32\, b_4B_1^5 - 16b_4^2\, B_1^4 + 2b_4^3B_1^3 + b_2^2 + 9b_4^2B_1^4 + 16B_1^6$$

$$+ 4b_3^2\, B_1^2 + 6b_3b_4B_1^2$$

$$- 8\, b_2B_1^3 - 4\, b_2b_3\, B_1 + 16\, b_3B_1^4 - 24\, b_4B_1^5 - 12\, b_3b_4\, B_1^3 = 0$$

$$B_1^6\,(-32 - 48 + 16) + B_1^5\,(+ 24\, b_4 + 32\, b_4 + 8\, b_4 + 24b_4 + 32b_4 -24\, b_4)$$

$$+ B_1^4\,(-16\, b_3 -16b_3 -6b_4^2 -8b_4^2 -16b_3 -3b_4^2 -16b_4^2 + 9b_4^2 + 16b_3 - 24b_4^2)$$

$$+ B_1^3\,(+12\, b_3b_4 + 8b_2 + 16b_3b_4 + 4b_3b_4 + 4b_3b_4 + 6b_4^3 + 8b_3b_4 -8b_2 -12b_3b_4 + 2b_4^3)$$

$$+ B_1^2\,(+16b_1 - 8b_3^2 -8b_2b_4 -3b_3b_4^2 -2b_2b_4 -4b_3b_4^2 - b_3b_4^2 + 4b_3^2 + 6b_2b_4)$$

$$+ B_1\,(-8b_1b_4 + 4b_2b_3 + 2b_3^2\, b_4 + 2b_2b_4^2 -4b_2b_3)$$

$$+ (+b_1b_4^2 - b_2b_3b_4 + b_2^2) = 0$$

$$B_1^6\,(-64) + B_1^5(96\, b_4) + B_1^4\,(-48\, b_4^2 -32b_3) + B_1^3\,(+32\, b_3b_4 + 8b_4^3)$$

$$B_1^2\,(+16\, b_1 - 4\, b_3^2 -4b_2b_4 -8b_3b_4^2) + B_1\,(-8b_1b_4 + 2b_3^2\, b_4 + 2b_2b_4^2)$$

$$+ (b_1b_4^2 - b_2b_3b_4 + b_2^2\,) = 0$$

Since $0 < B_1 < \frac{b_4}{2}$ and $B_1 \neq \frac{b_4}{4}$

let $B_1 = c\, b_4$ where $0 < c < \frac{1}{4}$ and/or $\frac{1}{4} < c < \frac{1}{2}$ (B121)

$B_1^2 = c^2 b_4^2$ $B_1^3 = c^3 b_4^3$ $B_1^4 = c^4 b_4^4$ $B_1^5 = c^5 b_4^5$ $B_1^6 = c^6 b_4^6$

$$c^6 (-64 b_4^6) + c^5 (+96\, b_4^6) + c^4 \left[b_4^4 (-48\, b_4^2 - 32 b_3) \right] + c^3 \left[b_4^3 (+32 b_3 b_4 + 8 b_4^3) \right]$$

$$+ c^2 \left[b_4^2 (+16\, b_1 - 4 b_3^2 - 4 b_2 b_4 - 8 b_3 b_4^2) \right] + c \left[b_4 (-8 b_1 b_4 + 2 b_3^2 b_4 + 2 b_2 b_4^2) \right]$$

$$+ (b_1 b_4^2 - b_2 b_3 b_4 + b_2^2) = 0$$

Try various values of c within both sets of limits indicated.

Let $w_6 = -64\, b_4^6$

$w_5 = +96\, b_4^6$ (B122)

$w_4 = b_4^4 (-48\, b_4^2 - 32\, b_3)$

$w_3 = b_4^3 (+32\, b_3 b_4 + 8 b_4^3)$

$w_2 = b_4^2 (+16\, b_1 - 4 b_3^2 - 4 b_2 b_4 - 8 b_3 b_4^2)$

$w_1 = b_4 (-8 b_1 b_4 + 2 b_3^2\, b_4 + 2 b_2 b_4^2)$

$w_0 = (b_1 b_4^2 - b_2 b_3 b_4 + b_2^2)$ (B123)

Then

$$w_6 c^6 + w_5 c^5 + w_4 c^4 + w_3 c^3 + w_2 c^2 + w_1 c + w_o = 0$$

$0 < c < \frac{1}{4}$ and $\frac{1}{4} < c < \frac{1}{2}$

Let $\alpha_3 = B_1$

$\beta_3 = A_1$

$\alpha_4 = B_2$

$\beta_4 = A_2$

Then

$$\alpha_3 = c\, b_4 \tag{B124}$$

$$\beta_3 = \sqrt{\frac{-b_2 - 3b_4\alpha_3^2 + 4\alpha_3^3 + 2b_3\alpha_3}{4\,\alpha_3 - b_4}} \tag{B125}$$

$$\alpha_4 = \frac{1}{2}(b_4 - 2\alpha_3) = \frac{1}{2}(b_4 - 2cb_4) = \frac{b_4}{2}(1 - 2c) \tag{B126}$$

$$\beta_4 = \sqrt{b_3 - (\beta_3^2 + \alpha_3^2) - 4\alpha_3\alpha_4 - \alpha_4^2} \tag{B127}$$

Partial Fraction Form

It has been assumed that the denominator of equation (B113) has the following form:

$$s\left[(s+\alpha_3)^2+\beta_3^2\right]\left[(s+\alpha_4)^2+\beta_4^2\right]$$

A solution of the roots of the denominator has been solved accordingly. See equations (B114), (B124), (B125), (B126), and (B127).

Therefore, from equation (B113), we must go to partial fractions.

$$\mathcal{L}\left[x\right] = \frac{a_4s^4+a_3s^3+a_2s^2+a_1s+a_0}{s^5+b_4s^4+b_3s^3+b_2s^2+b_1s} =$$

$$\frac{a_4s^4+a_3s^3+a_2s^2+a_1s+a_0}{s\left[(s+\alpha_3)^2+\beta_3^2\right]\left[(s+\alpha_4)^2+\beta_4^2\right]} =$$

$$\frac{A_1}{s}+\frac{A_3s+B_3}{\left[(s+\alpha_3)^2+\beta_3^2\right]}+\frac{A_4s+B_4}{\left[(s+\alpha_4)^2+\beta_4^2\right]} =$$

$$\frac{A_1}{s}+\frac{A_3s+B_3}{\left[s^2+2\alpha_3s+(\alpha_3^2+\beta_3^2)\right]}+\frac{A_4s+B_4}{\left[s^2+2\alpha_4s+(\alpha_4^2+\beta_4^2)\right]}$$

Put the last term in the form of a polynomial with a common denominator; it then becomes

$$
\begin{aligned}
&\Big\{ A_1 \left[s^2 + 2\alpha_3 s + (\alpha_3^2 + \beta_3^2)\right]\left[s^2 + 2\alpha_4 s + (\alpha_4^2 + \beta_4^2)\right] + A_3 s^2 \left[s^2 + 2\alpha_4 s + (\alpha_4^2 + \beta_4^2)\right] \\
&\quad + B_3 s \left[s^2 + 2\alpha_4 s + (\alpha_4^2 + \beta_4^2)\right] + A_4 s^2 \left[s^2 + 2\alpha_3 s + (\alpha_3^2 + \beta_3^2)\right] \\
&\quad + B_4 s \left[s^2 + 2\alpha_3 s + (\alpha_3^2 + \beta_3^2)\right]\Big\} \div \Big\{ \left[s\ s^2 + 2\alpha_3 s + (\alpha_3^2 + \beta_3^2)\right] \\
&\quad \left[s^2 + 2\alpha_4 s + (\alpha_4^2 + \beta_4^2\right]\Big\}
\end{aligned}
$$

The numerator becomes

$$
\begin{aligned}
&A_1 \Big[s^4 + 2\alpha_4 s^3 + s^2(\alpha_4^2 + \beta_4^2) + 2\alpha_3 s^3 + 4\alpha_3\alpha_4 s^2 + 2\alpha_3 s(\alpha_4^2 + \beta_4^2) + s^2(\alpha_3^2 + \beta_3^2) \\
&\qquad + 2\alpha_4(\alpha_3^2 + \beta_3^2)s \\
&\quad + (\alpha_3^2 + \beta_3^2)(\alpha_4^2 + \beta_4^2)\Big] + A_3 s^4 + 2\alpha_4 A_3 s^3 + A_3(\alpha_4^2 + \beta_4^2)s^2 + B_3 s^3 + 2\alpha_4 B_3 s^2 \\
&\quad + B_3(\alpha_4^2 + \beta_4^2)s + A_4 s^4 + 2\alpha_3 A_4 s^3 + A_4(\alpha_3^2 + \beta_3^2)s^2 + B_4 s^3 + 2B_4\alpha_3 s^2 \\
&\quad + B_4(\alpha_3^2 + \beta_3^2)s
\end{aligned}
$$

If coefficients of like terms are grouped, the numerator becomes

$$s^4 \left[A_1 + A_3 + A_4\right] + s^3 \left[2 \alpha_4 A_1 + 2\alpha_3 A_1 + 2\alpha_4 A_3 + B_3 + 2\alpha_3 A_4 + B_4\right]$$

$$+ s^2 \left[(\alpha_4^2 + \beta_4^2)A_1 + 4\alpha_3\alpha_4 A_1 + (\alpha_3^2 + \beta_3^2)A_1 + (\alpha_4^2+\beta_4^2)A_3 + 2\alpha_4 B_3 + (\alpha_3^2+\beta_3^2)A_4 + 2B_4\alpha_3\right]$$

$$+ s\left[2\alpha_3(\alpha_4^2+\beta_4^2)A_1 + 2\alpha_4(\alpha_3^2 + \beta_3^2)A_1 + (\alpha_4^2 + \beta_4^2)\, B_3 + (\alpha_3^2 + \beta_3^2)\, B_4\right]$$

$$+ \left[A_1\,(\alpha_3^2 + \beta_3^2)\,(\alpha_4^2 + \beta_4^2)\right]$$

This numerator must equal

$$a_4\, s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o$$

Therefore, coefficients of like terms may be equated.

$$a_4 = A_1 + A_3 + A_4 \tag{B128}$$

$$a_3 = 2\,\alpha_4 A_1 + 2\,\alpha_3 A_1 + 2\alpha_4 A_3 + B_3 + 2\alpha_3 A_4 + B_4 \tag{B129}$$

$$a_2 = (\alpha_4^2+\beta_4^2)A_1 + 4\alpha_3\alpha_4 A_1 + (\alpha_3^2 + \beta_3^2)A_1 + (\alpha_4^2+\beta_4^2)A_3 + 2\alpha_4 B_3$$

$$+ (\alpha_3^2 + \beta_3^2)\, A_4 + 2\, B_4\alpha_3 \tag{B130}$$

$$a_1 = 2\alpha_3(\alpha_4^2 + \beta_4^2)A_1 + 2\alpha_4(\alpha_3^2 + \beta_3^2)A_1 + (\alpha_4^2 + \beta_4^2)B_3 + (\alpha_3^2 + \beta_3^2)\, B_4 \tag{B131}$$

$$a_o = (\alpha_3^2 + \beta_3^2)\,(\alpha_4^2 + \beta_4^2)\, A_1 \tag{B132}$$

From equation (B132)

$$A_1 = \frac{a_o}{(\alpha_3^2 + \beta_3^2)(\alpha_4^2 + \beta_4^2)} \qquad \text{(B133)}$$

Let $g_3 = (\alpha_3^2 + \beta_3^2) \qquad g_4 = (\alpha_4^2 + \beta_4^2)$ (B134)

Then $A_1 = \frac{a_o}{(g_3)(g_4)}$

Equation (B128) becomes

$$A_3 + A_4 = a_4 - A_1 = d_1$$

Equation (B129) becomes

$$2\alpha_4 A_3 + 2\alpha_3 A_4 + B_3 + B_4 = a_3 - 2A_1(\alpha_3 + \alpha_4) = d_2$$

Equation (B130) becomes

$$g_4 A_3 + g_3 A_4 + 2\alpha_4 B_3 + 2\alpha_3 B_4 = a_2 - A_1(g_4 + g_3 + 4\alpha_3\alpha_4) = d_3$$

Equation (B131) becomes

$$g_4 B_3 + g_3 B_4 = a_1 - A_1(2\alpha_3 g_4 + 2\alpha_4 g_3) = d_4$$

In equation (B135),

$$\text{let } c_{1_x} = A_1 = \frac{a_{o_x}}{(g_3)(g_4)} \qquad \text{(B136)}$$

Then

$$d_{1_x} = a_{4_x} - c_{1_x} \tag{E137}$$

$$d_{2_x} = a_{3_x} - c_{1_x} b_4 \tag{E138}$$

$$d_{3_x} = a_{2_x} - c_{1_x} b_3 \tag{E139}$$

$$d_{4_x} = a_{1_x} - c_{1_x} b_2 \tag{E140}$$

Solve these four simultaneous equations using matrices

$$\begin{bmatrix} (1) & (1) & (0) & (0) & d_1 \\ (2\alpha_4) & 2\alpha_3 & 1 & 1 & d_2 \\ g_4 & g_3 & 2\alpha_4 & 2\alpha_3 & d_3 \\ 0 & 0 & g_4 & g_3 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & d_1 \\ (2\alpha_4 - 2\alpha_4) & (2\alpha_3 - 2\alpha_4) & 1 & 1 & (d_2 - 2\alpha_4 d_1) \\ (g_4 - g_4) & (g_3 - g_4) & 2\alpha_4 & 2\alpha_3 & (d_3 - g_4 d_1) \\ 0 & 0 & g_4 & g_3 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & d_1 \\ 0 & 1 & \frac{1}{(2\alpha_3 - 2\alpha_4)} & \frac{1}{(2\alpha_3 - 2\alpha_4)} & \frac{(d_2 - 2\alpha_4 d_1)}{(2\alpha_3 - 2\alpha_4)} \\ 0 & (g_3 - g_4) & 2\alpha_4 & 2\alpha_3 & (d_3 - g_4 d_1) \\ 0 & 0 & g_4 & g_3 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{(2\alpha_3-2\alpha_4)} & \frac{-1}{(2\alpha_3-2\alpha_4)} & d_1 - \frac{(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} \\ 0 & 1 & \frac{1}{(2\alpha_3-2\alpha_4)} & \frac{1}{(2\alpha_3-2\alpha_4)} & \frac{(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} \\ 0 & 0 & 2\alpha_4 - \frac{(g_3-g_4)}{(2\alpha_3-2\alpha_4)} & 2\alpha_3 - \frac{(g_3-g_4)}{(2\alpha_3-2\alpha_4)} & (d_3-g_4 d_1) - \frac{(g_3-g_4)(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} \\ 0 & 0 & g_4 & g_3 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \left[\frac{-1}{(2\alpha_3-2\alpha_4)} + \frac{g_3}{g_4(2\alpha_3-2\alpha_4)}\right] & \left[d_1 - \frac{(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} + \frac{d_4}{g_4(2\alpha_3-2\alpha_4)}\right] \\ 0 & 1 & 0 & \left[\frac{1}{(2\alpha_3-2\alpha_4)} - \frac{g_3}{g_4(2\alpha_3-2\alpha_4)}\right] & \left[\frac{(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} - \frac{d_4}{g_4(2\alpha_3-2\alpha_4)}\right] \\ 0 & 0 & 0 & \left[2\alpha_3 \frac{-(g_3-g_4)}{(2\alpha_3-2\alpha_4)} \frac{-g_3}{g_4}(2\alpha_4) + \frac{g_3}{g_4} \frac{(g_3-g_4)}{(2\alpha_3-2\alpha_4)}\right] & \left[(d_3-g_4 d_1) \frac{-(g_3-g_4)(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} - \frac{d_4}{g_4}(2\alpha_4) + \frac{d_4}{g_4} \frac{(g_3-g_4)}{(2\alpha_3-2\alpha_4)}\right] \\ 0 & 0 & 1 & \frac{g_3}{g_4} & \frac{d_4}{g_4} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & \text{[illegible]} & \text{[illegible]} \\ 0 & 0 & 0 & \text{[illegible]} & \text{[illegible]} \\ 0 & 1 & 0 & \text{[illegible]} & \text{[illegible]} \\ 1 & 0 & 0 & \text{[illegible]} & \text{[illegible]} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \\ 0 & 0 & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \\ 0 & 1 & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \\ 1 & 0 & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \end{bmatrix}$$

$$\left[\begin{array}{ccccl} A_3 & A_4 & B_3 & B_4 & \\ 1 & 0 & 0 & 0 & \left[d_1 \frac{-(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} + \frac{d_4}{g_4(2\alpha_3-2\alpha_4)}\right] - B_4\left[\frac{-1}{(2\alpha_3-2\alpha_4)} + \frac{g_3}{g_4(2\alpha_3-2\alpha_4)}\right] \\ 0 & 1 & 0 & 0 & \left[\frac{(d_2-2\alpha_4 d_1)}{(2\alpha_3-2\alpha_4)} - \frac{d_4}{g_4(2\alpha_3-2\alpha_4)}\right] - B_4\left[\frac{1}{(2d_3-2\alpha_4)} - \frac{g_3}{g_4(2\alpha_3-2\alpha_4)}\right] \\ 0 & 0 & 0 & 1 & \qquad\qquad B_4 \\ 0 & 0 & 1 & 0 & \qquad \left[\frac{d_4}{g_4} - \frac{g_3}{g_4} B_4\right] \end{array}\right]$$

$$B_4 = \frac{\left[(d_3-g_4 d_1) - \frac{(g_3-g_4)(d_2-2\alpha_4 d_1)}{(2\alpha_3 - 2\alpha_4)} - \frac{d_4}{g_4}(2\alpha_4) + \frac{d_4}{g_4}\frac{(g_3-g_4)}{(2\alpha_3-2\alpha_4)}\right]}{\left[2\alpha_3 \frac{-(g_3-g_4)}{(2\alpha_3-2\alpha_4)} - \frac{g_3}{g_4}(2\alpha_4) + \frac{g_3}{g_4}\frac{(g_3-g_4)}{(2\alpha_3 - 2\alpha_4)}\right]} = c_{6x}$$

$$B_4 = \frac{(2\alpha_3-2\alpha_4)(g_4)d_3-g_4 d_1) - (g_4)(g_3-g_4)(d_2-2\alpha_4 d_1) - 2\alpha_4 d_4(2\alpha_3-2\alpha_4) + d_4(g_3-g_4)}{2\alpha_3(2\alpha_3-2\alpha_4)(g_4) - g_4(g_3-g_4) - 2\alpha_4 g_3(2\alpha_3-2\alpha_4) + g_3(g_3-g_4)}$$

$$q^{1} = \frac{[illegible]}{[illegible]}$$

$$\frac{\left[[illegible] \right]}{q^{2} = \left[[illegible] \right]} = a^{[illegible]}$$

0	0	1	0	[illegible]	
0	0	0	1	[illegible]	
0	1	0	0	[illegible]	[illegible]
1	0	0	0	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]		

$$c_{6x} = B_{4x} = \frac{2(\alpha_3-\alpha_4)\left[g_4 d_{3x} - g_4^2 d_{1x} - 2\alpha_4 d_{4x}\right] + (g_3-g_4)\left[-g_4 d_{2x} + 2\alpha_4 g_4 d_{1x} + d_{4x}\right]}{2(\alpha_3-\alpha_4)\left[2\alpha_3 g_4 - 2\alpha_4 g_3\right] + (g_3-g_4)\left[(g_3 - g_4)\right]} \qquad \text{(B141)}$$

$$c_{5x} = B_{3x} = \frac{d_{4x} - g_3 B_{4x}}{g_4} = \frac{d_{4x} - g_3 c_{6x}}{g_4}$$

$$c_{4x} = A_{4x} = \left[\frac{g_4(d_{2x} - 2\alpha_4 d_{1x})}{2\, g_4(\alpha_3-\alpha_4)}\right] - B_{4x}\left[\frac{g_4 - g_3}{2g_4(\alpha_3-\alpha_4)}\right] = \frac{g_4(d_{2x}-2\alpha_4 d_{1x}) - d_{4x} + B_{4x}(g_3-g_4)}{2\, g_4(\alpha_3 - \alpha_4)} \qquad \text{(B143)}$$

$$c_{3x} = A_{3x} = d_{1x} - A_{4x} = d_{1x} - C_{4x} \qquad \text{(B144)}$$

$$[x] = \frac{A_1}{s} + \frac{A_3 s + B_3}{\left[(s+\alpha_3)^2 + \beta_3^2\right]} + \frac{A_4 s + B_4}{\left[(s+\alpha_4)^2 + \beta_4^2\right]} =$$

$$\frac{A_1}{s} + \frac{(s+\alpha_3)\,\phi_{23} + \beta_3 \phi_{13}}{\beta_3\left[(s+\alpha_3)^2 + \beta_3^2\right]} + \frac{(s+\alpha_4)\,\phi_{24} + \beta_4\,\phi_{14}}{\beta_4\left[(s+\alpha_4)^2 + \beta_4^2\right]} \qquad \text{(B145)}$$

where, according to (36),

$$\phi_{23} = \beta_3 A_3 \qquad \phi_{24} = \beta_4 A_4$$

$$\phi_{13} = B_3 - \alpha_3 A_3 \qquad \phi_{14} = B_4 - \alpha_4 A_4 \qquad \text{(B146)}$$

$$x = A_1 + \frac{1}{\beta_3} e^{-\alpha_3 t} (\phi_{23} \cos \beta_3 t + \phi_{13} \sin \beta_3 t) + \frac{1}{\beta_4} e^{-\alpha_4 t} (\phi_{24} \cos \beta_4 t + \phi_{14} \sin \beta_4 t) \tag{B147}$$

$$P_{23x} = (\beta_3)(c_{3x}) \qquad P_{24x} = (\beta_4)(c_{4x})$$

$$P_{13x} = (c_{5x}) - (\alpha_3)(c_{3x}) \qquad P_{14x} = (c_{6x}) - (\alpha_4)(c_{4x}) \tag{B148}$$

$$x = c_{1x} + \frac{1}{\beta_3} e^{-\alpha_3 t} (P_{23x} \cos \beta_3 t + P_{13x} \sin \beta_3 t) + \frac{1}{\beta_4} e^{-\alpha_4 t} (P_{24x} \cos \beta_4 t + P_{14x} \sin \beta_4 t) \tag{B149}$$

$$\dot{x} = \frac{-\alpha_3}{\beta_3} e^{-\alpha_3 t} (P_{23x} \cos \beta_3 t + P_{13x} \sin \beta_3 t)$$

$$+ e^{-\alpha_3 t} (-P_{23x} \sin \beta_3 t + P_{13x} \cos \beta_3 t)$$

$$\frac{-\alpha_4}{\beta_4} e^{-\alpha_4 t} (P_{24x} \cos \beta_4 t + P_{14x} \sin \beta_4 t)$$

$$+ e^{-\alpha_4 t} (-P_{24x} \sin \beta_4 t + P_{14x} \cos \beta_4 t) \tag{B150}$$

$$\ddot{x} = \frac{+\alpha_3^2}{\beta_3} e^{-\alpha_3 t} (P_{23x} \cos \beta_3 t + P_{13x} \sin \beta_3 t)$$

$$-2 \alpha_3 e^{-\alpha_3 t} (-P_{23x} \sin \beta_3 t + P_{13x} \cos \beta_3 t)$$

$(r_1 \sin [illegible] + r_2 \cos [illegible])^{[illegible]} [illegible] (r_2 \cos [illegible] + r_1 \sin [illegible])^{[illegible]} [illegible]$

[illegible]

[illegible]

[illegible]

$(r_1 \sin [illegible] + r_2 \cos [illegible])^{[illegible]} = \frac{[illegible]}{c^2} + (r_2 \sin [illegible] + r_1 \cos [illegible])^{[illegible]} [illegible]$

[illegible]

$(r_2 \sin [illegible] + r_1 \cos [illegible])^{[illegible]} = [illegible]$

$(r_1 \cos [illegible] - r_2 \sin [illegible])^{[illegible]} [illegible]$

$(r_2 \sin [illegible] + r_1 \cos [illegible])^{[illegible]} = \frac{[illegible]}{c^2}$

[illegible] $(r_1 \cos [illegible] + r_2 \sin [illegible])^{[illegible]} = [illegible]$

$(r_1 \sin [illegible] + r_2 \cos [illegible])^{[illegible]} = \frac{[illegible]}{c^2} [illegible]$

$(r_2 \cos [illegible] + r_2 \sin [illegible])^{[illegible]} [illegible]$

$$+ \beta_3 e^{-\alpha_3 t} (-P_{23x} \cos \beta_3 t - P_{13x} \sin \beta_3 t)$$

$$+ \frac{(\alpha_4^2 - \beta_4^2)}{\beta_4} e^{-\alpha_4 t} (P_{24x} \cos \beta_4 t + P_{14x} \sin \beta_4 t)$$

$$- 2\alpha_4 e^{-\alpha_4 t} (-P_{24x} \sin \beta_4 t + P_{14x} \cos \beta_4 t)$$

$$\ddot{x} = \frac{(\alpha_3^2 - \beta_3^2)}{\beta_3} e^{-\alpha_3 t} (P_{23x} \cos \beta_3 t + P_{13x} \sin \beta_3 t)$$

$$-2 \alpha_3 e^{-\alpha_3 t} (-P_{23x} \sin \beta_3 t + P_{13x} \cos \beta_3 t)$$

$$+ \frac{(\alpha_4^2 - \beta_4^2)}{\beta_4} e^{-\alpha_4 t} (P_{24x} \cos \beta_4 t + P_{14x} \sin \beta_4 t)$$

$$- 2 \alpha_4 e^{-\alpha_4 t} (-P_{24x} \sin \beta_4 t + P_{14x} \cos \beta_4 t) \tag{E151}$$

The solution for z is as follows:

$$\mathcal{L}[z] = \frac{N_2}{\text{Den.}} = \frac{\begin{array}{ccc} (a_{11}s^2 + b_{11}s) & (d_{11} + d_{12}s + \frac{d_{13}}{s}) & (c_{13}) \\ (b_{21}s + c_{21}) & (d_{21} + d_{22}s + \frac{d_{23}}{s}) & (a_{23}s^2 + b_{23}s + c_{23}) \\ (c_{31}) & (\frac{d_{33}}{s}) & (c_{33}) \end{array}}{D_4 s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0}$$

$$N_2 = a_{11}d_{21}c_{33}s^2 + a_{11}d_{22}c_{33}s^3 + a_{11}d_{23}c_{33}s + b_{11}d_{21}c_{33}s + b_{11}d_{22}c_{33}s^2 + b_{11}d_{23}c_{33}$$

$$+ d_{11}a_{23}c_{31}s^2 + d_{11}b_{23}c_{31}s + d_{11}c_{23}c_{31} + d_{12}a_{23}c_{31}s^3 + d_{12}b_{23}c_{31}s$$

$$+ d_{13}a_{23}c_{31}s + d_{13}b_{23}c_{31} + d_{13}c_{23}c_{31}\left(\frac{1}{s}\right) + c_{13}b_{21}d_{33} + c_{13}c_{21}d_{33}\left(\frac{1}{2}\right)$$

$$- a_{11}a_{23}d_{33}s^3 - a_{11}b_{23}d_{33}s^2 - a_{11}c_{23}d_{33}s - b_{11}a_{23}d_{33}s^2 - b_{11}d_{23}d_{33}s - b_{11}c_{23}d_{33}$$

$$- d_{11}b_{21}c_{33}s - d_{11}c_{21}c_{33} - d_{12}b_{21}c_{33}s^2 - d_{12}c_{21}c_{33}s - d_{12}b_{21}c_{33} - d_{13}c_{21}c_{33}\left(\frac{1}{s}\right)$$

$$- c_{13}d_{21}c_{31} - c_{13}d_{22}c_{31}s - c_{13}d_{23}c_{31}\left(\frac{1}{s}\right)$$

$$N_2 = N_{23}s^3 + N_{22}s^2 + N_{21}s + N_{20} + N_{19}\left(\frac{1}{s}\right) \qquad \text{(H52)}$$

$$N_{23} = (a_{11}d_{22}c_{33} + d_{12}a_{23}c_{31} - a_{11}a_{23}d_{33})$$

$$N_{22} = (a_{11}d_{21}c_{33} + b_{11}d_{22}c_{33} + d_{11}a_{23}c_{31} + d_{12}b_{23}c_{31} - a_{11}b_{23}d_{33} - b_{11}a_{23}d_{33} - d_{12}b_{21}c_{33})$$

$$N_{21} = (a_{11}d_{23}c_{33} + b_{11}d_{21}c_{33} + d_{11}b_{23}c_{31} + d_{12}c_{23}c_{31} + d_{13}a_{23}c_{31} - a_{11}c_{23}d_{33} - b_{11}b_{23}d_{33}$$

$$-d_{11}b_{21}c_{33} - d_{12}c_{21}c_{33} - c_{13}d_{22}c_{31})$$

$$N_{20} = (b_{11}d_{23}c_{33} + d_{11}c_{23}c_{31} + d_{13}b_{23}c_{31} + c_{13}b_{21}d_{33} - b_{11}c_{23}d_{33} - d_{11}c_{21}c_{33} - d_{13}b_{21}c_{33}$$

$$-c_{13}d_{21}c_{31})$$

$$N_{19} = (d_{13}c_{23}c_{31} + c_{13}c_{21}d_{33} - d_{13}c_{21}c_{33} - c_{13}d_{23}c_{31})$$

$$L\,[z] = \frac{N_{19} + N_{20}s + N_{21}s^2 + N_{22}s^3 + N_{23}s^4}{s\,(D_0 + D_1 s + D_2 s^2 + D_3 s^3 + D_4 s^4} \qquad \text{(B153)}$$

Let $a_{4z} = \frac{N_{23}}{D_4}$ $a_{3z} = \frac{N_{22}}{D_4}$ $a_{2z} = \frac{N_{21}}{D_4}$ $a_{1z} = \frac{N_{20}}{D_4}$ $a_{0z} = \frac{N_{19}}{D_4}$

where [illegible]

$$\Gamma\,[x] = \frac{[illegible]}{[illegible]}$$ ([illegible])

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

From (B112),

$$b_4 = \frac{D_3}{D_4} \quad b_3 = \frac{D_2}{D_4} \quad b_2 = \frac{D_1}{D_4} \quad b_1 = \frac{D_0}{D_4} \quad b_o = 0$$

Then,

$$\mathcal{L}\left[z\right] = \frac{a_{4z}s^4 + a_{3z}s^3 + a_{2z}s^2 + a_{1z}s + a_{0z}}{s^5 + b_4 s^4 + b_3 s^3 + b_2^2 + b_1^2}$$

It is noted that this is the same as equation (B113) except the constant coefficients (a_{4z}, a_{3z}, etc.) of the numerator have different values. The solution for z is the same as for x except for that change.

$$c_{1z} = \frac{a_{0z}}{(g_3)(g_4)} \tag{B156}$$

$$d_{1z} = a_{4z} - c_{1z} \tag{B157}$$

$$d_{2z} = a_{3z} - (c_{1z})(b_4)$$

$$d_{3z} = a_{2z} - (c_{1z})(b_3)$$

$$d_{4z} = a_{1z} - (c_{1z})(b_2) \tag{B158}$$

$$c_{6z} = B_{4z} = \frac{2(\alpha_3 - \alpha_4)\left[g_4 d_{3z} - g_4^2 d_{1z} - 2\alpha_4 d_{4z}\right] + (g_3 - g_4)\left[-g_4 d_{2z} + 2\alpha_4 g_4 d_{1z} + d_{4z}\right]}{2\,(\alpha_3 - \alpha_4)\left[2\alpha_3 g_4 - 2\alpha_4 g_3\right] + (g_3 - g_4)^2}$$

$$c_{5z} = B_{3z} = \frac{d_{4z} - g_3\, B_{4z}}{g_4} = \frac{d_{4z} - g_3 c_{6z}}{g_4} \tag{B159}$$

$$c_{4z} = A_{4z} = \frac{g_4(d_{2z} - 2\alpha_4 d_{1z}) - d_{4z} + c_{6z}\,(g_3 - g_4)}{2\, g_4\, (\alpha_3 - \alpha_4)} \tag{B160}$$

$$c_{3z} = A_{3z} = d_{1z} - A_{4z} = d_{1z} - c_{4z} \tag{B161}$$

$$P_{23z} = (\beta_3)(c_{3z}) \tag{B162}$$

$$P_{13z} = (c_{5z}) - (\alpha_3)(c_{3z})$$

$$P_{24z} = (\beta_4)(c_{4z})$$

$$P_{14z} = (c_{6z}) - (\alpha_4)(c_{4z})$$

$$z = c_{1z} + \frac{1}{\beta_3} e^{-\alpha_3 t} (P_{23z} \cos \beta_3 t + P_{13z} \sin \beta_3 t)$$

$$+ \frac{1}{\beta_4} e^{-\alpha_4 t} (P_{24z} \cos \beta_4 t + P_{14z} \sin \beta_4 t) \tag{B163}$$

$$\dot{z} = \frac{-\alpha_3}{\beta_3}\; e^{-\alpha_3 t} (P_{23z} \cos \beta_3 t + P_{13z} \sin \beta_3 t)$$

$$+ e^{-\alpha_3 t} (-P_{23z} \sin \beta_3 t + P_{13z} \cos \beta_3 t)$$

$$\frac{-\alpha_4}{\beta_4} e^{-\alpha_4 t} (P_{24z} \cos \beta_4 t + P_{14z} \sin \beta_4 t)$$

$$+ e^{-\alpha_4 t} (-P_{24z} \sin \beta_4 t + P_{14z} \cos \beta_4 t) \tag{B164}$$

$$\ddot{z} = \frac{(\alpha_3^2 - \beta_3^2)}{\beta_3} e^{-\alpha_3 t} (P_{23z} \cos \beta_3 t + P_{13z} \sin \beta_3 t)$$

$$-2 \alpha_3 e^{-\alpha_3 t} (-P_{23z} \sin \beta_3 t + P_{13z} \cos \beta_3 t)$$

$$\frac{+(\alpha_4^2 - \beta_4^2)}{\beta_4} e^{-\alpha_4 t} (P_{24z} \cos \beta_4 t + P_{14z} \sin \beta_4 t)$$

$$-2 \alpha_4 e^{-\alpha_4 t} (-P_{24z} \sin \beta_4 t + P_{14z} \cos \beta_4 t) \tag{B165}$$

The solution for θ is as follows:

$$\mathcal{L}\,\boxed{\theta}\,\frac{N_3}{\text{Den.}} = \frac{\begin{vmatrix} (a_{11}s^2 + b_{11}s) & (a_{12}s^2 + b_{12}s + c_{12}) & (d_{11} + d_{12}s + \frac{d_{13}}{s}) \\ (b_{21}s + c_{21}) & (a_{22}s^2 + b_{22}s + c_{22}) & (d_{21} + d_{22}s + \frac{d_{23}}{s}) \\ (c_{31}) & (c_{32}) & (\frac{d_{33}}{s}) \end{vmatrix}}{D_4 s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0}$$

$$N_3 = a_{11}a_{22}d_{33}s^3 + a_{11}b_{22}d_{33}s^2 + a_{11}c_{22}d_{33}s + b_{11}a_{22}d_{33}s^2 + b_{11}b_{22}d_{33}s + b_{11}c_{22}d_{33}$$

$$+ a_{12}d_{21}c_{31}s^2 + a_{12}d_{22}c_{31}s^3 + a_{12}d_{23}c_{31}s + b_{12}d_{21}c_{31}s + b_{12}d_{22}c_{31}s^2 + b_{12}d_{23}c_{31}$$

$$+ c_{12}d_{21}c_{31} + c_{12}d_{22}c_{31}s + c_{12}d_{23}c_{31}(\frac{1}{s}) + d_{11}b_{21}c_{32}s + d_{11}c_{21}c_{32} + d_{12}b_{21}c_{32}s^2$$

$$+ d_{12}c_{21}c_{32}s + d_{13}b_{21}c_{32} + d_{13}c_{21}c_{32}(\frac{1}{s}) - a_{11}d_{21}c_{32}s^2 - a_{11}d_{22}c_{32}s^3 - a_{11}d_{23}c_{32}s$$

$$- b_{11}d_{21}c_{32}s - b_{11}d_{22}c_{32}s^2 - b_{11}d_{23}c_{32} - a_{12}b_{21}d_{33}s^2 - a_{12}c_{21}d_{33}s - b_{12}b_{21}d_{33}s$$

$$- b_{12}c_{21}d_{33} - c_{12}b_{21}d_{33} - c_{12}c_{21}d_{33}(\frac{1}{s}) - d_{11}a_{22}c_{31}s^2 - d_{11}b_{22}c_{31}s - d_{11}c_{22}c_{31}$$

$$- d_{12}a_{22}c_{31}s^3 - d_{12}b_{22}c_{31}s^2 - d_{12}c_{22}c_{31}s - d_{13}a_{22}c_{31}s - d_{13}b_{22}c_{31} - d_{13}c_{22}c_{31}(\frac{1}{s})$$

$$N_3 = N_{33}s^3 + N_{32}s^2 + N_{31}s + N_{30} + N_{29}(\frac{1}{s}) \qquad (B166)$$

$a^{3} = a^{33}e_3 + a^{32}e_2 +$ [illegible] $\left(\frac{a}{r}\right)$ [illegible]

$-$ [illegible] $\left(\frac{a}{r}\right)$

[illegible] $\left(\frac{a}{r}\right) -$ [illegible]

$-$ [illegible]

$+$ [illegible] $\left(\frac{a}{r}\right) -$ [illegible]

$+$ [illegible] $\left(\frac{a}{r}\right) +$ [illegible]

$+$ [illegible]

$a^{2} =$ [illegible]

$$N_{33} = (a_{11}a_{22}d_{33} + a_{12}d_{22}c_{31} - a_{11}d_{22}c_{32} - d_{12}a_{22}c_{31})$$

$$N_{32} = (a_{11}b_{22}d_{33} + b_{11}a_{22}d_{33} + a_{12}d_{21}c_{31} + b_{12}d_{22}c_{31} + d_{12}b_{21}c_{32} - a_{11}d_{21}c_{32} - b_{11}d_{22}c_{32}$$

$$-a_{12}b_{21}d_{33} - d_{11}a_{22}c_{31} - d_{12}b_{22}c_{31})$$

$$N_{31} = (a_{11}c_{22}d_{33} + b_{11}b_{22}d_{33} + a_{12}d_{23}c_{31} + b_{12}d_{21}c_{31} + c_{12}d_{22}c_{31} + d_{11}b_{21}c_{32} + d_{12}c_{21}c_{32}$$

$$-a_{11}d_{23}c_{32} - b_{11}d_{21}c_{32} - a_{12}c_{21}d_{33} - b_{12}b_{21}d_{33} - d_{11}b_{22}c_{31} - d_{12}c_{22}c_{31} - d_{13}a_{22}c_{31})$$

$$N_{30} = (b_{11}c_{22}d_{33} + b_{12}d_{23}c_{31} + c_{12}d_{21}c_{31} + d_{11}c_{21}c_{32} + d_{13}b_{21}c_{32} - b_{11}d_{23}c_{32} - b_{12}c_{21}d_{33}$$

$$-c_{12}b_{21}d_{33} - d_{11}c_{22}c_{31} - d_{13}b_{22}c_{31})$$

$$N_{29} = (c_{12}d_{23}c_{31} + d_{13}c_{21}c_{32} - c_{12}c_{21}d_{33} - d_{13}c_{22}c_{31})$$

$g^{33} =$ [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\mathcal{L}\,[\theta] = \frac{N_{29} + N_{30}s + N_{31}s^2 + N_{32}s^3 + N_{33}s^4}{s\ (D_0 + D_1 s + D_2 s^2 + D_3 s^3 + D_4 s^4} \qquad \text{(B167)}$$

Let $a_{40} = \dfrac{N_{33}}{D_4}$ $\quad a_{30} = \dfrac{N_{32}}{D_4}$ $\quad a_{20} = \dfrac{N_{31}}{D_4}$ $\quad a_{10} = \dfrac{N_{30}}{D_4} = a_{00} = \dfrac{N_{29}}{D_4}$

Then,

$$\mathcal{L}\,[\theta] = \frac{a_{40}s^4 + a_{30}s^3 + a_{20}s^2 + a_{10}s + a_{00}}{s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s} \qquad \text{(B169)}$$

It is noted that this is the same as equations (B113) and (B155) except for the constants in the numerator. Therefore the solution is as follows:

$$c_{10} = \frac{a_{00}}{(g_3)(g_4)} \qquad \text{(B170)}$$

$$d_{10} = a_{40} - c_{10} \qquad \text{(B171)}$$

$$d_{20} = a_{30} - (c_{10})(b_4)$$

$$d_{30} = a_{20} - (c_{10})(b_3)$$

$$d_{40} = a_{10} - (c_{10})(b_2)$$

$$c_{60} = \frac{2(\alpha_3-\alpha_4)\left[g_4 d_{30} - g_4^2 d_{10} - 2\alpha_4 d_{40}\right] + (g_3-g_4)\left[-g_4 d_{20} + 2\alpha_4 g_4 d_{10} + d_{40}\right]}{2\,(\alpha_3 - \alpha_4)\left[2\alpha_3 g_4 - 2\alpha_4 g_3\right] + (g_3 - g_4)^2}$$

$$c_{50} = \frac{d_{40} - g_3 c_{60}}{g_4} \qquad \text{(B173)}$$

$$c_{40} = \frac{g_4(d_{20} - 2\alpha_4 d_{10}) - d_{40} + c_{60}\,(g_3 - g_4)}{2\,g_4\,(\alpha_3 - \alpha_4)} \qquad \text{(B174)}$$

$$c_{30} = d_{10} - c_{40} \qquad \text{(B175)}$$

$$P_{230} = (\beta_3)(c_{30}) \qquad \text{(B176)}$$

$$P_{130} = (c_{50}) - (\alpha_3)(c_{30})$$

$$P_{240} = (\beta_4)(c_{40})$$

$$P_{140} = (c_{60}) - (\alpha_4)(c_{40})$$

$$\theta = c_{10} + \frac{1}{\beta_3} e^{-\alpha_3 t}\,(P_{230}\cos\beta_3 t + P_{130}\sin\beta_3 t)$$

$$+ \frac{1}{\beta_4} e^{-\alpha_4 t}\,(P_{240}\cos\beta_4 t + P_{140}\sin\beta_4 t) \qquad \text{(B177)}$$

$$\dot{\theta} = \frac{-\alpha_3}{\beta_3} e^{-\alpha_3 t} (P_{230} \cos \beta_3 t + P_{130} \sin \beta_3 t)$$

$$+e^{-\alpha_3 t} (-P_{230} \sin \beta_3 t + P_{130} \cos \beta_3 t)$$

$$\frac{-\alpha_4}{\beta_4} e^{-\alpha_4 t} (P_{240} \cos \beta_4 t + P_{140} \sin \beta_4 t)$$

$$+ e^{-\alpha_4 t} (-P_{240} \sin \beta_4 t + P_{140} \cos \beta_4 t) \qquad \text{(B178)}$$

$$\ddot{\theta} = \frac{(\alpha_3^2 - \beta_3^2)}{\beta_3} e^{-\alpha_3 t} (P_{230} \cos \beta_3 t + P_{130} \sin \beta_3 t)$$

$$- 2 \alpha_3 e^{-\alpha_3 t} (-P_{230} \sin \beta_3 t + P_{130} \cos \beta_3 t)$$

$$+ \frac{(\alpha_4^2 - \beta_4^2)}{\beta_4} e^{-\alpha_4 t} (P_{240} \cos \beta_4 t + P_{140} \sin \beta_4 t)$$

$$- 2 \alpha_4 e^{-\alpha_4 t} (-P_{240} \sin \beta_4 t + P_{140} \cos \beta_4 t) \qquad \text{(B179)}$$

Now there is a complete description of the motion during the sliding phase.

It is noted that x_2, $\dot{x}_2$, z_2, $\dot{z}_2$, θ_2 and $\dot{\theta}_2$ are used as initial conditions for the LaPlace equations for sliding. This is in addition to their use in linearizing.

However, $\ddot{x}_2$, $\ddot{z}_2$ and $\ddot{\theta}_2$ are used only for linearization (and not for initial conditions). It is not appropriate to use the State 2 conditions of accelerations for initial conditions for sliding. For that reason it is considered, for use in the sliding equations, to consider

$$\ddot{x}_2 = -1.0 \qquad \text{(B180)}$$

$$\ddot{z}_2 = 0.0$$

$$\ddot{\theta}_2 = 0.0$$

Vertical Force on Bow During Sliding

It is important to note that the downward force, F_{BZS}, may be solved for each time t during the sliding phase. (This value should not exceed $(F_{BZC})_2$ since this would imply local crushing should have started again.)

The value of F_{BZS} may be determined from equation (B92).

$$F_{BZS} = - T_{BOL}\theta + \frac{T_{BOL}}{v_1}\left(\frac{dx}{dt}\right)\theta - T_f\, z$$

$$- T_f\,(LCG-LCF)\theta - k_h\left(\frac{dz}{dt}\right) - m_z\,\frac{d^2z}{dt^2}$$

where θ may be obtained from equation (B177)

$\frac{dx}{dt}$ (B150)

z (B163)

$\frac{dz}{dt}$ (B164)

$\frac{d^2z}{dt^2}$ (B165)

Termination of Sliding Phase, State 3

If the equations of velocity (B150), (B164), and (B178) combined, the velocity of a point on the bow in contact with the ice may be determined. When this velocity becomes zero the sliding phase has terminated, State 3. Using this time, t_3, x_3, z_3, and θ_3 can be determined from (B149), (B163) and (B177) and this will give us the location on the ship of the point of ice support. Using this point the static equilibrium problem may be solved (presuming slipping does not start immediately) and the downward force under the bow may be determined.

From Figure B-XIX it may be seen that the velocity of point A on the bow may be expressed in terms of the z-component

$$v_{Az} = \frac{dz}{dt} - \frac{d\theta}{dt}(GA)_x \qquad \text{(B181)}$$

and the x-component

$$v_{Ax} = \frac{dx}{dt} - \frac{d\theta}{dt}(GA)_z \qquad \text{(B182)}$$

Since these two components are related as shown in Figure B-XIX, the value of one may be used to determine the value of the other. However, when $v_{Az} = 0$ then $v_{Ax} = 0$. For that reason either equation (B181) or (B182) may be used to find the time, t_3, for $v_A = 0$.

Equation (B182) for the velocity component of A in the x-direction will be used.

$$v_{Ax} = \frac{dx}{dt} - \frac{d\Theta}{dt} (GA_z)$$

From equation (B86)

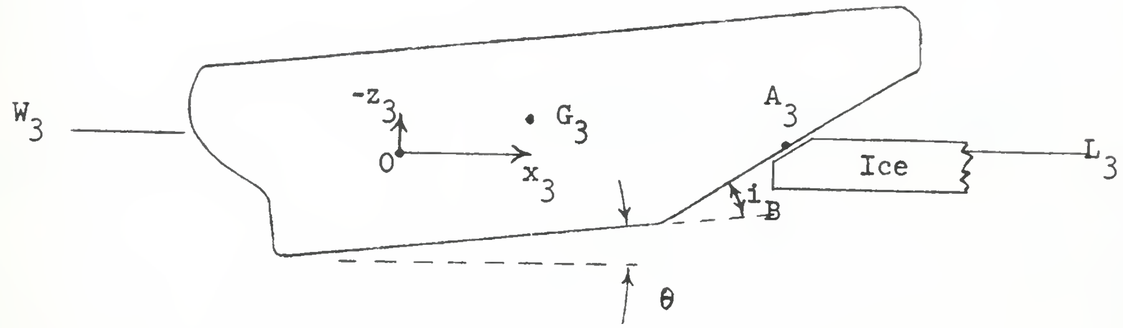
$$(GA)_z = (H-KG) + z$$

$$v_{Ax} = \frac{dx}{dt} - \frac{d\Theta}{dt} \left[(H-KG) + z \right] \qquad \text{(B183)}$$

The values to be used come from equations (B150), (B178), and (B163) respectively. For each substitution of t into these equations and then the substitution of these values into equation, a velocity v_{Ax} results. When v_{Ax} becomes zero the sliding phase has terminated and State 3 has been reaches.

Figure B-XXVI

Illustration of Position at State 3



From equation (B88),

$$(GA_3)_x = k_4 - x_3$$

From equation (B86)

$$(GA_3)_z = (H-KG) + z_3$$

State 4

It seems quite probable that when there is no velocity of the bow relative to the ice that all velocities will be zero, or negligible. However, the static equilibrium problem should be solved regardless based on support of the bow at point A_3 for time t_3.

It should also be noted that the friction force due to sliding has now disappeared.

Based on Figure B-XXVII Newton's equations may be applied for static equilibrium.

$$\sum F_x = 0$$

$$T_{BOL} \cos \theta_4 - F_{BX4} = 0 \qquad \text{(B184)}$$

$$\sum F_z = 0$$

$$-F_{BZ4} - T_{BOL} \sin \theta_4 - \Delta - T_f h_4 + \Delta = 0$$

$$-F_{BZ4} - T_{BOL} \sin \theta_4 - T_f h_4 = 0 \qquad \text{(B185)}$$

It is to be noted that F_{BZ4} will be of greater magnitude if the bollard thrust is eliminated (stopping the screws) as long as static equilibrium can be maintained by static friction at the bow.

It may be seen in Figure B-XXVIII that the change in draft at the center of gravity from State 3 to State 4 is equal to

$$z_4 - z_3 = (GA_3)_x \, (\theta_4 - \theta_3)$$

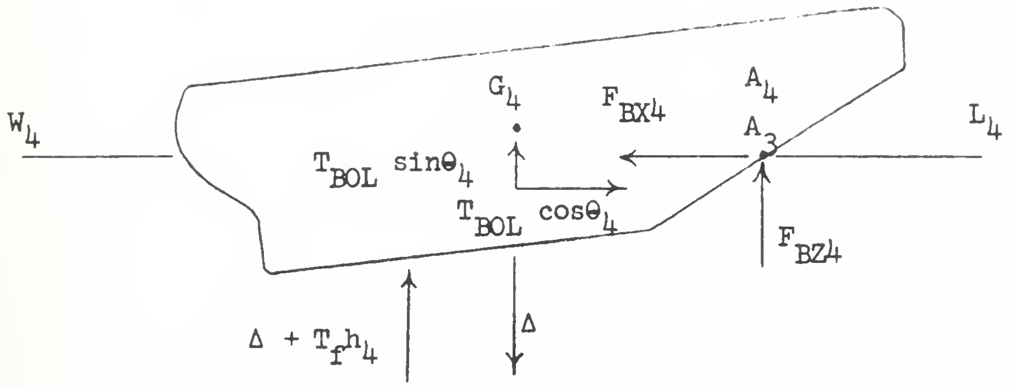
Therefore

$$z_4 = z_3 + (GA_3)_x \, (\theta_4 - \theta_3) \qquad \text{(B186)}$$

From equation (B17)

Figure B-XXVII

Free Body Diagram for Static Equilibrium, State 4



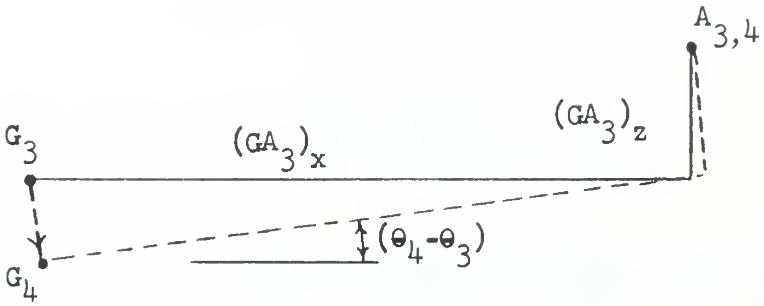
h ≡ increase in draft at LCF

T_f ≡ pounds per foot immersion

T_{BOL} ≡ bollard thrust, pounds

Figure B-XXVIII

Illustration of Moment Arms $(GA_4)_x$ and $(GA_4)_z$



$$(G_4A_4)_x = (GA_3)_x \cos(\theta_4-\theta_3) - (GA_3)_z \sin(\theta_4-\theta_3)$$

and for small angle $(\theta_4-\theta_3)$

$$(G_4A_4)_x = (GA_3)_x - (GA_3)_z (\theta_4 - \theta_3)$$

and

$$(G_4A_4)_z = (GA_3)_z + (GA_3)_x (\theta_4-\theta_3)$$

$$h = z + (LCG - LCF)\,\theta$$

$$h_4 = z_4 + (LCG - LCF)\,\theta_4$$

From equation (B186)

$$h_4 = z_3 + (LCG - LCF)\,\theta_4 + (GA_3)_x\,(\theta_4 - \theta_3)$$

$$\sum M_{\curvearrowleft} = 0$$

$$F_{BZ4}\,(G_4A_4)_x + F_{BX4}\,(G_4A_4)_z + T_{BOL}\,(KG-d)$$

$$- (\Delta + T_f h_4)\,GM_L\,\theta_4 = 0 \qquad \text{(B187)}$$

$$F_{BZ4}\left[(GA_3)_x - (GA_3)_z\,(\theta_4 - \theta_3)\right] + F_{BX4}\left[(GA_3)_z + (GA_3)_x(\theta_4 - \theta_3)\right]$$

$$+ T_{BOL}\,(KG-d) - \Delta\,GM_L\,\theta_4 - T_f\,GM_L\,\theta_4\left[z_3 + (LCG - LCF)\,\theta_4 + (GA_3)_x\,(\theta_4 - \theta_3)\right]$$

$$= 0 \qquad \text{(B188)}$$

It may be seen that three unknowns (θ_4, F_{BX4}, and F_{BZ4}) appear in the three equations (B184), (B185), and (B188).

As mentioned before, the maximum static vertical force F_{BZ4} can be attained when the bollard thrust is eliminated. (Under this condition it

is assumed that static friction at the bow is sufficient to maintain equilibrium or at least the acceleration sliding back down the ice is negligible.)

Set $T_{BOL} = 0$ by stopping thrust.

Then $F_{BX4} = 0$

Equations (B185) and (B188) may now be written as follows:

$$-F_{BZ4} - T_f z_3 - T_f (LCG-LCF)\theta_4 - T_f (GA_3)_x (\theta_4 - \theta_3) = 0 \qquad (B189)$$

$$F_{BZ4}\left[(GA_3)_x - (GA_3)_z (\theta_4 - \theta_3)\right] - \Delta\, GM_L\, \theta_4 \qquad (B190)$$

$$- T_f\, GM_L\, \theta_4 \left[z_3 + (LCG - LCF)\theta_4 + (GA_3)_x (\theta_4 - \theta_3)\right] = 0 \qquad (B190)$$

Expand equation (B189).

$$-F_{BZ4} - T_f z_3 - T_f (LCG-LCF)\,\theta_4 - T_f (GA_3)_x \theta_4$$

$$+ T_f (GA_3)_x\, \theta_3 = 0$$

$$-F_{BZ4} - T_f z_3 + T_f (GA_3)_x \theta_3 = T_f \left[(LCG - LCF) + (GA_3)_x\right] \theta_4$$

$$\theta_4 = \frac{-F_{BZ4} - T_f z_3 + T_f\,(GA_3)_x\theta_3}{T_f\left[(LCG - LCF) + (GA_3)_x\right]}$$

$$\theta_4 = \frac{\frac{-F_{BZ4}}{T_f} - z_3 + (GA_3)_x\theta_3}{(LCG - LCF) + (GA_3)_x} \qquad \text{(B191)}$$

Set $d_1 = (LCG - LCF) + (GA_3)_x$ (B192)

$$\theta_4 = \frac{-F_{BZ4}}{d_1\,T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x\theta_3}{d_1} \qquad \text{(B193)}$$

Substitute equation (B193) into equation (B190)and solve for F_{BZ4}.

$$F_{BZ4}\,(GA_3)_x + F_{BZ4}\,(GA_3)_z\,\theta_3$$

$$-F_{BZ4}\,(GA_3)_z\left[\frac{-F_{BZ4}}{d_1\,T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x\,\theta_3}{d_1}\right]$$

$$-\Delta\,GM_L\left[\frac{-F_{BZ4}}{d_1\,T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x\,\theta_3}{d_1}\right]$$

$$T_f\,GM_L\,z_3\left[\frac{-F_{BZ4}}{d_1\,T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x\,\theta_3}{d_1}\right]$$

$$- T_f \; GM_L \; (LCG - LCF) \left[\frac{- F_{BZ4}}{d_1 \; T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x \; \theta_3}{d_1} \right]^2$$

$$+ T_f \; GM_L \; (GA_3)_x \; \theta_3 \left[\frac{- F_{BZ4}}{d_1 \; T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x \; \theta_3}{d_1} \right]$$

$$- T_f \; GM_L \; (GA_3)_x \left[\frac{- F_{BZ4}}{d_1 \; T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x \; \theta_3}{d_1} \right]^2$$

$$= 0$$

$$\theta_4^2 = \left[\frac{- F_{BZ4}}{d_1 \; T_f} - \frac{z_3}{d_1} + \frac{(GA_3)_x \; \theta_3}{d_1} \right]^2$$

$$\theta_4^2 = \frac{F_{BZ4}^2}{d_1^2 \; T_f^2} + \frac{F_{BZ4} \; z_3}{d_1^2 \; T_f} - \frac{F_{BZ4} \; (GA_3)_x \; \theta_3}{d_1^2 \; T_f}$$

$$+ \frac{z_3^2}{d_1^2} + \frac{F_{BZ4} \; z_3}{d_1^2 \; T_f} - \frac{(GA_3)_x \; z_3 \; \theta_3}{d_1^2}$$

$$+ \frac{(GA_3)_x^2 \; \theta_3^2}{d_1^2} \quad \frac{-(GA_3)_x z_3 \; \theta_3}{d_1^2} - \frac{- F_{BZ4} \; (GA_3)_x \; \theta_3}{d_1^2 \; T_f}$$

$$\theta_4^2 = \frac{F_{BZ4}^2}{d_1^2\ T_f^2} + \frac{z_3^2}{d_1^2} + \frac{(GA_3)_x^2\ \theta_3^2}{d_1^2} + \frac{2\ F_{BZ4}\ z_3}{d_1^2\ T_f}$$

$$- \frac{2\ F_{BZ4}\ (GA_3)_x\ \theta_3}{d_1^2\ T_f} - \frac{2\ (GA_3)_x\ z_3\ \theta_3}{d_1^2}$$

$$F_{BZ4} (GA_3)_x + F_{BZ4} (GA_3)_z \Theta_3 + F_{BZ4}^2 \frac{(GA_3)_z}{d_1 T_f} + F_{BZ4} \frac{(GA_3)_z z_3}{d_1} - F_{BZ4} \frac{(GA_3)_z (GA_3)_x \Theta_3}{d_1}$$

$$+ F_{BZ4} \frac{\Delta GM_L}{d_1 T_f} + \frac{\Delta GM_L z_3}{d_1} - \frac{\Delta GM_L (GA_3)_x \Theta_3}{d_1} + \frac{F_{BZ4} GM_L z_3}{d_1} + \frac{T_f GM_L z_3^2}{d_1}$$

$$- \frac{T_f GM_L z_3 (GA_3)_x \Theta_3}{d_1} - F_{BZ4} \frac{GM_L (GA_3)_x \Theta_3}{d_1} - \frac{T_f GM_L (GA_3)_x \Theta_3 z_3}{d_1} + \frac{(GA_3)_x^2 \Theta_3^2}{d_1}$$

$$- F_{BZ4}^2 \frac{GM_L (LCG-LCF)}{d_1^2 T_f} - \frac{T_f GM_L (LCG - LCF) z_3^2}{d_1^2} - \frac{T_f GM_L (LCG - LCF) (GA_3)_x^2 \Theta_3^2}{d_1^2}$$

$$- F_{BZ4} \frac{(2) GM_L (LCG - LCF) z_3}{d_1^2} + F_{BZ4} \frac{(2) GM_L (LCG - LCF) (GA_3)_x \Theta_3}{d_1^2} + \frac{2 T_f GM_L (LCG-LCF)(GA_3)_x z_3 \Theta_3}{d_1^2}$$

$$- F_{BZ4}^{2} \frac{GM_L (GA_3)_x}{d_1^2 T_f} - \frac{T_f GM_L (GA_3)_x z_3^2}{d_1^2} - \frac{T_f GM_L (GA_3)_x^3 \theta_3^2}{d_1^2} - F_{BZ4} \frac{(2) GM_L (GA_3)_x z_3}{d_1^2}$$

$$+ F_{BZ4} \frac{(2) GM_L (GA_3)_x^2 \theta_3}{d_1^2} + \frac{2 T_f GM_L (GA_3)_x^2 z_3 \theta_3}{d_1^2} = 0$$

$$+ F_{BZ4}^{2} \left[\frac{(GA_3)_z}{d_1 T_f} - \frac{GM_L (LCG - LCF)}{d_1^2 T_f} - \frac{GM_L (GA_3)_x}{d_1^2 T_f} \right]$$

$$+ F_{BZ4} \left[(GA_3)_x + (GA_3)_z \theta_3 + \frac{(GA_3)_z z_3}{d_1} - \frac{(GA_3)_z (GA_3)_x \theta_3}{d_1} + \frac{\Delta GM_L}{d_1 T_f} + \frac{GM_L z_3}{d_1} \right.$$

$$- \frac{GM_L (GA_3)_x \theta_3}{d_1} - \frac{2 GM_L (LCG-LCF) z_3}{d_1^2} + \frac{2 GM_L (LCG-LCF)(GA_3)_x \theta_3}{d_1^2}$$

$$- \frac{q^2}{\omega^2 \, (\sigma^2)^2 \kappa^2} - \frac{q_5^2}{\kappa \, \omega^2 \, (\ldots)\kappa^2} + \frac{q^2}{\kappa \, \omega^2 \, (\ldots)(\sigma^2)\, \kappa^2}$$

[illegible]

$$+ z^{[illegible]} \left[(\omega^2)^2 + (\omega^2)^2 \kappa^2 - \frac{q^2}{(\omega^2)^2 \kappa^2} - \frac{q^2}{(\omega^2)^2 (\omega^2)^2 \kappa^2} + \frac{q^2}{\kappa \, \omega^2} + \frac{q^2}{\omega^2 \, \kappa^2} \right.$$

$$+ z^{[illegible]} \left[\frac{q^2 \kappa^4}{(\omega^2)^2} - \frac{q_5^2 \, \kappa^2}{\omega^2 \, (\ldots - \ldots)} + \frac{q^2 \, \kappa^2}{\omega^2 \, (\omega^2)^2} \right]$$

$$+ z^{[illegible]} \frac{q_5^2}{(2)\, \omega^2 \, (\omega^2)^2 \kappa^2} + \frac{q^2}{\kappa \, \omega^2 \, (\omega^2)^2 \kappa^2} = 0$$

$$- z^{[illegible]} \frac{q_5^2 \kappa^4}{\omega^2 \, (\omega^2)^2} - \frac{q_5^2}{\kappa \, \omega^2 \, (\omega^2)^2 \kappa_5^2} - \frac{q_5^2}{\kappa \, \omega^2 \, (\omega^2)^2 \kappa^2} - z^{[illegible]} \frac{q^2}{(1)\, \omega^2 \, (\omega^2)^2 \kappa^2}$$

$$- \frac{2\ GM_L\ (GA_3)_x z_3}{d_1^2} + \frac{2\ GM_L\ (GA_3)_x^2\ \theta_3}{d_1^2} \Bigg]$$

$$+ \Bigg[\frac{\Delta\ GM_L\ z_3}{d_1} - \frac{\Delta\ GM_L\ (GA_3)_x \theta_3}{d_1} + \frac{T_f\ GM_L\ z_3^2}{d_1} - \frac{T_f\ GM_L\ z_3\ (GA_3)_z\ \theta_3}{d_1} - \frac{T_f GM_L\ (GA_3)_x\ \theta_3 z_3}{d_1}$$

$$+ \frac{T_f\ GM_L\ (GA_3)_x^2\ \theta_3^2}{d_1} - \frac{T_f GM_L\ (LCG\text{-}LCF) z_3^2}{d_1^2} - \frac{T_f GM_L\ (LCG - LCF)\ (GA_3)_x^2\ \theta_3^2}{d_1^2}$$

$$+ \frac{2\ T_f\ GM_L\ (LCG\text{-}LCF)(GA_3)_x z_3 \theta_3}{d_1^2} - \frac{T_f GM_L\ (GA_3)_x z_3^2}{d_1^2} - \frac{T_f GM_L\ (GA_3)_x^3\ \theta_3^2}{d_1^2} + \frac{2\ T_f GM_L\ (GA_3)_x^2 z_3 \theta_3}{d_1^2} \Bigg]$$

$$= 0$$

Therefore

$$a_4\ F_{BZ4}^2 + b_4\ F_{BZ4} + c_4 = 0 \qquad (B194)$$

$$[illegible] = 0 \qquad ([illegible])$$

Therefore

$$= 0$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

where

$$a_4 = \frac{+(GA_3)_z}{d_1\,T_f} \quad \frac{-\,GM_L\,(LCG - LCF)}{d_1^2\,T_f} \quad \frac{-\,GM_L\,(GA_3)_x}{d_1^2\,T_f}$$

$$b_4 = (GA_3)_x + (GA_3)_z\theta_3 \quad \frac{+(GA_3)_z z_3}{d_1} - \frac{(GA_3)_z (GA_3)_x \theta_3}{d_1} \frac{+\Delta\,GM_L}{d_1 T_f} + \frac{GM_L z_3}{d_1} \quad \frac{-\,GM_L\,(GA_3)_x \theta_3}{d_1}$$

$$\frac{-\,2\,GM_L\,(LCG - LCF) z_3}{d_1^2} + \frac{2\,GM_L (LCG - LCF)(GA_3)_x \theta_3}{d_1^2} \quad \frac{-\,2\,GM_L (GA_3)_x z_3}{d_1^2} + \frac{2\,GM_L (GA_3)_x^2\,\theta_3}{d_1^2}$$

$$c_4 = \frac{\Delta\,GM_L\,z_3}{d_1} - \frac{\Delta\,GM_L (GA_3)_x \theta_3}{d_1} + \frac{T_f GM_L\,z_3^2}{d_1} - \frac{T_f GM_L z_3 (GA_3)_x \theta_3}{d_1} \quad - \frac{T_f GM_L (GA_3)_x \theta_3 z_3}{d_1}$$

$$+ \frac{T_f GM_L\,(GA_3)_x^2 \theta_3^2}{d_1} \quad \frac{-T_f GM_L}{d_1^2} \left[(LCG - LCF)\,z_3^2 + (LCG - LCF)(GA_3)_x^2\,\theta_3^2 - 2(LCG\text{-}LCF)(GA_3)_x z_3 \theta_3 \right.$$

$$\left. + (GA_3)_x\,z_3^2 + (GA_3)_x^3\,\theta_3^2 - 2\,(GA_3)_x^2\,z_3\,\theta_3 \right]$$

$$+ (\sigma r^3)^{2}\, [illegible] + (\sigma r^3)_3^2\, e_S^3 - 3\,(\sigma r^3)_S^2\, [illegible] \Big]$$

$$[illegible] = \frac{\sigma^T}{[illegible]\,(\sigma r^3)_S^2 e_S^3}\;\frac{\sigma_S^T}{[illegible]}\Big[(r\sigma_2 - r\sigma_1)\, e_S^3 + (r\sigma_2 - r\sigma_1)(\sigma r^3)_S^2\, e_S^3 - 2(r\sigma_2 - r\sigma_1)(\sigma r^3)^{2}\, [illegible]$$

$$\sigma^P = \frac{\sigma^T}{[illegible]} - \frac{\sigma^T}{[illegible]\,(\sigma r^3)^2\, [illegible]} + \frac{\sigma^T}{[illegible]\, e_S^3} - \frac{\sigma^T}{[illegible]\,(\sigma r^3)^2\, [illegible]} - \frac{\sigma^T}{[illegible]\,(\sigma r^3)^2\, [illegible]}$$

$$-\frac{\sigma_S^T}{[illegible]\,(r\sigma_2 - r\sigma_1)\, [illegible]} + \frac{\sigma_S^T}{[illegible]\,(r\sigma_2 - r\sigma_1)(\sigma r^3)^2\, [illegible]} - \frac{\sigma_S^T}{[illegible]\,(\sigma r^3)^2\, e^3} - \frac{\sigma_S^T}{[illegible]}$$

$$p^t = (\sigma r^3)^{2} + (\sigma r^3)^{2}\, [illegible]\;\frac{\sigma^T}{[illegible]\,(\sigma r^3)^2\, [illegible]} - \frac{\sigma^T}{(\sigma r^3)^2\,(\sigma r^3)^2\, [illegible]}\;\frac{[illegible]}{[illegible]} + \frac{\sigma^T}{[illegible]} - \frac{\sigma^T}{[illegible]\,(\sigma r^3)^2\, [illegible]}$$

$$\sigma^t = \frac{\sigma_S^T\, [illegible]}{[illegible]\,(\sigma r^3)^2}\;\frac{\sigma_S^T\, [illegible]}{[illegible]\,(r\sigma_2 - r\sigma_1)}\;\frac{\sigma_S^T\, [illegible]}{[illegible]\,(\sigma r^3)^2}$$

$$F_{BZ4} = \frac{-\,b_4 \pm \sqrt{b_4^2 - 4a_4c_4}}{2\,a_4}$$

It is presumed the values are real and unequal (i.e. $b_4^2 - 4a_4c_4 \quad 0$) and that the larger of these values is the significant one. Therefore the static force under the bow is

$$F_{BZ4} = \frac{-\,b_4 + \sqrt{b_4^2 - 4a_4c_4}}{2\,a_4} \qquad \text{(B195)}$$

The final trim (θ_4), may be obtained from equation (B193),

$$\theta_4 = \frac{-F_{BZ4}}{d_1 T_f} - \frac{z_3}{d_1} + \frac{(GA)_{3x}\theta_3}{d_1}$$

The change of position in the x-direction, on settling, is negligible. The final value for z (z_4) may be obtained from equation (B186),

$$z_4 = z_3 + (GA_3)_x\,(\theta_4 - \theta_3)$$

Extracting Thrust

Once the icebreaker has completely ceased moving (State 4) it frequently becomes stuck in that position due to static friction which can be significantly greater than kinetic friction.

It is possible,indeed probable, that an icebreaker designed to attain a high downward sustained force may also, unfortunately, require a very large backing thrust to remove itself. For that reason it is important to know what backing thrust will be required to back down. This will be called "extracting thrust",

It is necessary to create enough extracting thrust to overcome the friction between the bow plating and the ice. The direction of this friction force is parallel to the stem since that is the direction of impending motion at the bow.

The values of x_4, z_4, $(GA)_{z4}$, $(GA)_{x4}$, and Θ_4 are known and valid for this condition since changing them would imply the icebreaker is not held by static friction.

Refer to Figure B-XXIX. The force normal to the bow plating on each side is N/2. The friction force is then f_s N/2.

As may be seen in Figure B-XXX the force normal to the stem, in the centerline plane, may be expressed as

$$N \cos \beta$$

where β = angle between normal to plating and centerline plane.

These forces may be resolved into a vertical component and a horizontal component respectively. See Figure B-XXX.

Figure B-XXIX

Forces on Bow When Backing is Impending

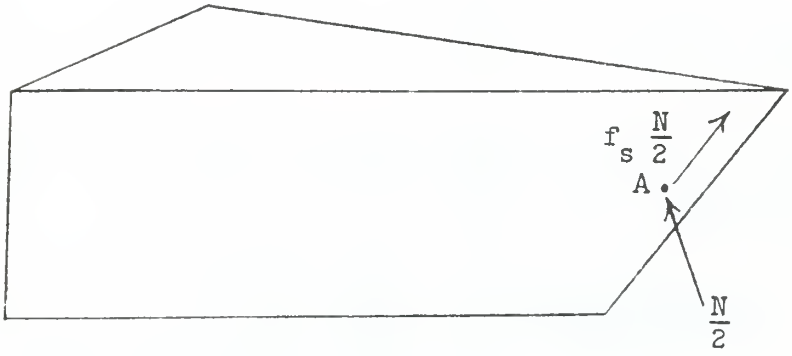
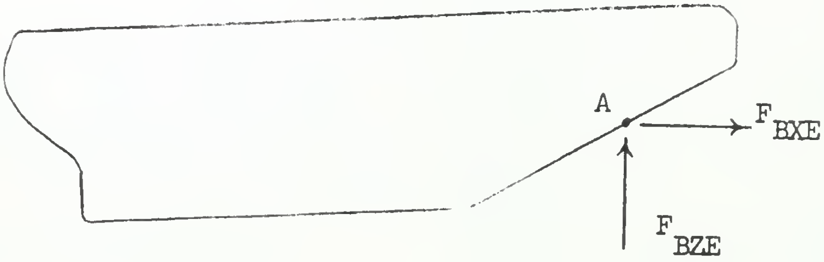


Figure B-XXX

Component Bow Forces When Backing is Impending



$$F_{BZE} = (N \cos \beta) \cos (i_B + \Theta_4) + f_s N \sin (i_B + \Theta_4)$$

$$F_{BZE} = N \left[(\cos \beta) \cos (i_B + \Theta_4) + f_s \sin (i_B + \Theta_4) \right] \qquad \text{(B196)}$$

$$F_{BXE} = - (N \cos \beta) \sin (i_B + \Theta_4) + f_s N \cos (i_B + \Theta_4)$$

$$F_{BXE} = N \left[- (\cos \beta) \sin (i_B + \Theta_4) + f_s \cos (i_B + \Theta_4) \right] \qquad \text{(B197)}$$

Summing forces in the z-direction, (See Figure B-XXXI)

$$\Delta - (\Delta + T_f h_4) + E_t \sin \Theta_4 - F_{BZE} = 0$$

As may be realized from equation (B189),

$$\Delta - (\Delta + T_f h_4) = + F_{BZ4}$$

Therefore,

$$+ F_{BZ4} + E_t \sin \Theta_4 - F_{BZE} = 0$$

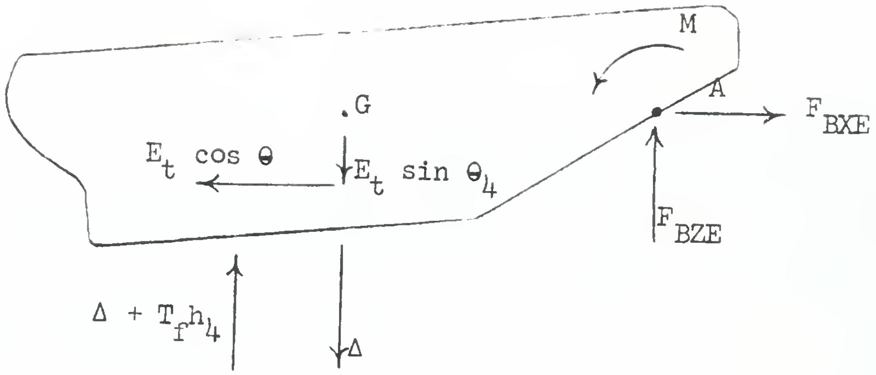
$$F_{BZE} = F_{BZ4} + E_t \sin \Theta_4 \qquad \text{(B198)}$$

Summing forces in the x-direction,

$$F_{BXE} - E_t \cos \Theta_4 = 0 \qquad \text{(B199)}$$

Figure B-XXXI

Free Body Diagram for Extraction



Let $a_7 = (\cos \beta) \cos (i_B + \theta_4) + f_s \sin (i_B + \theta_4)$ (B200)

and $b_7 = - (\cos \beta) \sin (i_B + \theta_4) + f_s \cos (i_B + \theta_4)$ (B201)

Then equation (B198 becomes

$$a_7 N = F_{BZ4} + E_t \sin \theta_4$$

$$N = \frac{1}{a_7} (F_{BZ4} + E_t \sin \theta_4)$$

Equation (B199) becomes

$$b_7 N = E_t \cos \theta_4$$

$$N = E_t \frac{\cos \theta_4}{b_7}$$

Equating these equations

$$\frac{F_{BZ4}}{a_7} + E_t \frac{\sin \theta_4}{a_7} = N = E_t \frac{\cos \theta_4}{b_7}$$

$$\frac{F_{BZ4}}{a_7} = E_t \frac{\cos \theta_4}{b_7} - E_t \frac{\sin \theta_4}{a_7}$$

$$\frac{F_{BZ4}}{a_7} = E_t \left(\frac{\cos \theta_4}{b_7} - \frac{\sin \theta_4}{a_7}\right)$$

$$E_t = \frac{F_{BZ4}}{a_7 \left(\frac{\cos \theta_4}{b_7} - \frac{\sin \theta_4}{d_7}\right.}$$

$$E_t = \frac{F_{BZ4}}{(\frac{a_7}{h_7} \cos \theta_4 - \sin \theta_4)} \qquad \text{(B202)}$$

It is noted that all values needed to solve E_t from equation (B202) are known.

There is a small moment created which will help free the icebreaker. So neglecting this is on the safe side. Furthermore, if the line of action of the thrust passes through the point of contact, this moment vanishes.

Computer Program

Naturally the solution of all the preceding equations would be quite tedious and there would be a high probability of error. This is compounded by the fact that there are three iterative solutions involved. Furthermore, one solution by itself would be of little value; comparisons are needed.

For the reasons mentioned the solution has been programmed in Fortran and carried out on an I.B.M. 7094 computer.*

The following is a listing of the input data which must be supplied:

BP	Length between perpendiculars, ft.
B	Beam at waterline, ft.
H	Mean draft, ft.
DIS	Displacement, lb.
BA	Bow angle (from base line to stem), radians
SA	"Spread angle complement (normal to bow plating with respect to centerline plane), radians
V1	Impact velocity, ft./sec.
AL	α, Waterplane coefficient, dimensionless.
CF	LCF, Longitudinal position of the center of flotation (- if aft of amidships, + if fd), ft.
CG	LCG, Longitudinal position of the center of gravity (- if aft of amidships, + if frd), ft.
GK	KG, Height of center of gravity above base line, ft.

---

* This work was done in part at the Computation Center at M.I.T., Cambridge, Massachusetts.

D	Height of thrust line above base line near center of gravity, ft.
TB	Bollard thrust which would be attained for rpm used during crushing and sliding, lbs.
GM	GM_L, Longitudinal metacentric height, ft.
FK	Ice/ship kinetic friction coeff., dimensionless.
FS	Ice/ship static friction coefficient, dimensionless.
S/G	Compressive failure stress of ice, lb/ft^2

```
   M4045-3564,FMS,TEST,5,5,5000,0 DYNAMIC ICEBREAKING R.M.WHITE
   XEQ
   DYNAMIC ICEBREAKING  R. M. WHITE
36 READ 5,BP,B,H,DIS,BA,SA,V1,AL,CF,CG,GK,D,TB,GM,FK,FS,SIG
 5 FORMAT (4F15.3/4F15.3/4F15.3/4F15.3/F15.3)
   PRINT 41, BP,B,H,DIS,BA,SA,V1,AL,CF,CG,GK,D,TB,GM,FK,FS,SIG
41 FORMAT (6H   BP=,F15.3,5H   B=,F15.3,5H   H=,F15.3,7H   DIS=,F15.3
  1/6H   BA=,F15.3,6H   SA=,F15.3,6H   V1=,F15.3,6H   AL=,F15.3/
  26H   CF=,F15.3,6H   CG=,F15.3,6H   GK=,F15.3,5H   D=,F15.3/6H   TB
  3=,F15.3,6H   GM=,F15.3,6H   FK=,F15.3,6H   FS=,F15.3/7H   SIG=,
  4F15.3//)
   XM = (3.36E-2)*DIS                                           (B66)
   ZM = (5.78E-2)*DIS                                           (B67)
   RG = 0.22*BP                                                 (B71)
   THM = (5.0E-2)*(RG**2)*DIS                                   (B68)
   DP = (1.76E-2)*DIS*BP**1.5                                   (B69)
   DH = (5.29E-1)*DIS/BP**0.5                                   (B70)
   TF = (64.2)*BP*B*AL
   SIBA = SINF(BA)
   COBA = COSF(BA)
   TABA = SIBA/COBA
   SISA = SINF(SA)
   COSA = COSF(SA)
   P1 = (SIG*TABA/SISA)*(SIBA*(COSA+FK*SISA)+FK*COBA)           (B33)
   P2 = (SIG*TABA/SISA)*(COBA*(COSA+FK*SISA)-FK*SIBA)           (B39)
   P3 = P2*(BP/2.-CG)+P1*(H-GK)                                 (B51)
   A1 = THM
   B1 = DP
   C1 = DIS*GM                                                  (B56)
   D1 = V1**2*P3
   AL1 = -B1/(2.*A1)
   DISC1 = 4.*C1/A1-(B1**2)/(A1**2)
   IF (DISC1) 11,2,2                                            (B57)
 2 BE1 = 0.5*SQRTF(DISC1)
   AA1 = (2.*D1/C1**2)*(A1-B1**2/C1)
   AA2 = (2.*D1/((C1**2)*BE1))*(B1-AL1*(A1-(B1**2)/C1))
   A2 = ZM
   B2 = DH                                                      (B46)
   C2 = TF
   D2 = -P2*V1**2
   AL2 = -B2/(2.*A2)
   DISC2 = 4.*C2/A2-(B2**2)/(A2**2)
   IF (DISC2) 12,3,3                                            (B61)
 3 BE2 = 0.5*SQRTF(DISC2)
   BB1 = (2.*D2/C2**2)*(A2-B2**2/C2)
   BB2 = (2.*D2/((C2**2)*BE2))*(B2-AL2*(A2-(B2**2)/C2))
   PRINT 4, XM,ZM,RG,THM,DP,DH,TF,P1,P2,A1,B1,C1,P3,D1,AL1,BE1,AA1,
  1AA2,A2,B2,C2,D2,AL2,BE2,BB1,BB2
 4 FORMAT (4E12.4/5E12.4/5E12.4/4E12.4/4E12.4/4E12.4//)
   T = -0.05
 1 T = T+0.05
   EAL1T = EXPF(AL1*T)
   COB1T = COSF(BE1*T)
   SIB1T = SINF(BE1*T)
   TH = EAL1T*(AA1*COB1T+AA2*SIB1T)+D1*(T**2)/C1-2.*B1*D1*T/(C1**2)
                                                                (B58)
```

```
     1-AA1
      THD = AL1*EAL1T*(AA1*COB1T+AA2*SIB1T)+EAL1T*(-AA1*BE1*SIB1T+AA2 (B55)
     1*BE1*COB1T)+2.*D1*T/C1-2.*B1*D1/C1**2
      THDD = (AL1**2-BE1**2)*EAL1T*(AA1*COB1T+AA2*SIB1T)+2.*AL1*BE1*  (B59)
     1EAL1T*(-AA1*SIB1T+AA2*COB1T)+2.*D1/C1
      EAL2T = EXPF(AL2*T)
      COB2T = COSF(BE2*T)
      SIB2T = SINF(BE2*T)
      Z =EAL2T*(BB1*COB2T+BB2*SIB2T)+D2*(T**2)/C2-2.*B2*D2*T/(C2**2)-BB1(B62)
      ZD = AL2*EAL2T*(BB1*COB2T+BB2*SIB2T)+EAL2T*(-BB1*BE2*SIB2T+BB2     (B63)
     1*BE2*COB2T)+2.*D2*T/C2-2.*B2*D2/C2**2
      ZDD = (AL2**2-BE2**2)*EAL2T*(BB1*COB2T+BB2*SIB2T)+2.*AL2*BE2*      (B64)
     1EAL2T*(-BB1*SIB2T+BB2*COB2T)+2.*D2/C2
      X = V1*(T-P1*T**3/(12.*XM))                                  (B41)
      XD = SQRTF(V1**2-2.*P1*X**3/(3.*XM))                         (B40)
      XDD = -P1*X**2/XM                                            (B39)
      FXC = P1*X**2                                                (B47)
      FZC = P2*X**2                                                (B48)
      GAX = (BP/2.-CG)-((H-GK)+(BP/2.-CG)/TABA)*TH+Z/TABA          (B20)
      GAZ = H-GK+Z                                                 (B16)
      TAGA = (GAX*THD-ZD)/(XD-GAZ*THD)                             (B65)
      DIF = SINF(BA+TH)/COSF(BA+TH)-TAGA
      PRINT 6,T,TH,THD,THDD,Z,ZD,ZDD,X,XD,XDD,FXC,FZC,TAGA,GAX,GAZ,DIF
    6 FORMAT (F11.5/3F11.5/3F11.5/3F11.5/2E12.5/4F11.5//)
      IF (XD) 38,38,37
   38 PRINT 39, FZC
      GO TO 36
   39 FORMAT (44H   SHIP STOPPED DURING CRUSHING PHASE, FZC2=,E12.5//)
   37 IF (DIF) 14,14,7
    7 TL = T
      THL = TH
      THDL = THD
      THDDL = THDD
      ZL = Z
      ZDL = ZD
      ZDDL = ZDD
      XL = X
      XDL = XD
      XDDL = XDD
      FXCL = FXC
      FZCL = FZC
      TAGAL = TAGA
      GAXL = GAX
      GAZL = GAZ
      DIFL = DIF
      GO TO 1
   14 TERP = DIFL/(DIFL-DIF)
      T2 = TL+TERP*(T-TL)
      TH2 = THL+TERP*(TH-THL)
      THD2 = THDL+TERP*(THD-THDL)
      THDD2 = THDDL+TERP*(THDD-THDDL)
      Z2 = ZL+TERP*(Z-ZL)
      ZD2 = ZDL+TERP*(ZD-ZDL)
      ZDD2 = ZDDL+TERP*(ZDD-ZDDL)
      X2 = XL+TERP*(X-XL)
```

```
      XD2 = XDL+TERP*(XD-XDL)
      XDD2 = XDDL+TERP*(XDD-XDDL)
      FXC2 = FXCL+TERP*(FXC-FXCL)
      FZC2 = FZCL+TERP*(FZC-FZCL)
      TAGA2 = TAGAL+TERP*(TAGA-TAGAL)
      GAX2 = GAXL+TERP*(GAX-GAXL)
      GAZ2 = GAZL+TERP*(GAZ-GAZL)
      DIF2 = DIFL+TERP*(DIF-DIFL)
   10 PRINT 15, T2,TH2,THD2,THDD2,Z2,ZD2,ZDD2,X2,XD2,XDD2,FXC2,FZC2,
     1TAGA2,GAX2,GAZ2,DIF2
   15 FORMAT (6H    T2=,F11.5/7H   TH2=,F11.5,8H   THD2=,F11.5,
     19H   THDD2=,F11.5/6H   Z2=,F11.5,7H   ZD2=,F11.5,8H   ZDD2=,F11.5/
     26H   X2=,F11.5,7H   XD2=,F11.5,8H   XDD2=,F11.5/8H   FXC2=,E12.5,
     38H   FZC2=,E12.5/9H   TAGA2=,F11.5,6H   GAX2=,F11.5,8H   GAZ2=,
     4F11.5,8H   DIF2=,F11.5//)
      GO TO 16
   11 PRINT 13,DISC1
   13 FORMAT (E12.4)
      GO TO 36
   12 PRINT 13,DISC2
      GO TO 36
C     ICEBREAKER SLIDING PHASE SOLUTION  R. M. WHITE
   16 AS = COSA*SIBA+FK*COBA                                      (B80)
      BS = COSA*COBA-FK*SIBA                                      (B81)
      XDD2 = -1.0
      ZDD2 = 0.0                                                  (B180)
      THDD2 = 0.0
      P4 = GAX2+X2                                                (B87)
      P5 = 1.+(AS/BS)**2                                          (B94)
      HGK = H-GK
      CGCF = CG-CF
      GKD = GK-D
      A11 = -XM
      B11 = -(TB/V1)*(1.+AS*TH2/BS+P5*TH2**2)
      C11 = 0.0
      A12 = ZM*(AS/BS+P5*TH2)                                     (B95)
      B12 = DH*(AS/BS+P5*TH2)
      C12 = TF*(AS/BS+P5*TH2)
      A13 = 0.0
      B13 = 0.0
      C13 = TB*(AS/BS-AS*XD2/(BS*V1)+2.*P5*TH2-2.*P5*XD2*TH2/V1)
     1+TF*(AS*CGCF/BS+P5*Z2+2.*P5*CGCF*TH2)+P5*DH*ZD2+P5*ZM*ZDD2
      D1 = -TB*(1.+AS*XD2*TH2/(BS*V1)-P5*TH2**2+2.*P5*XD2*TH2**2/V1)
     1+TF*(P5*Z2*TH2+P5*CGCF*TH2**2)+P5*DH*ZD2*TH2+P5*ZM*ZDD2*TH2
      A21 = 0.0
      B21 = TB*(P4*TH2/V1-TH2*X2/V1+AS*HGK*TH2/(BS*V1)+P5*HGK*TH2**2/V1
     1+AS*TH2*Z2/(BS*V1)+P5*Z2*TH2**2/V1-GKD/V1)
      C21 = TB*(TH2-XD2*TH2/V1)+TF*(Z2+CGCF*TH2)+DH*ZD2+ZM*ZDD2
      A22 = ZM*(-P4+X2-AS*HGK/BS-P5*HGK*TH2-AS*Z2/BS-P5*TH2*Z2)  (B100)
      B22 = DH*(-P4+X2-AS*HGK/BS-P5*HGK*TH2-AS*Z2/BS-P5*TH2*Z2)
      C22 = TF*(X2-AS*HGK/BS-P5*HGK*TH2-P4-2.*AS*Z2/BS-AS*CGCF*TH2/BS
     1-2.*P5*Z2*TH2-P5*CGCF*TH2**2-GM*TH2)+TB*(-AS*TH2/BS+AS*XD2*TH2/
     2(BS*V1)-P5*TH2**2+P5*XD2*TH2**2/V1)+DH*(-AS*ZD2/BS-P5*ZD2*TH2)
     3+ZM*(-AS*ZDD2/BS-P5*ZDD2*TH2)
      A23 = -THM
```

```
  B23 = -DP
  C23 = TB*(-P4+P4*XD2/V1+X2-XD2*X2/V1-AS*HGK/BS+AS*HGK*XD2/(BS*V1)
 1-2.*P5*HGK*TH2+2.*P5*HGK*XD2*TH2/V1-AS*Z2/BS+AS*XD2*Z2/(BS*V1)
 2-2.*P5*Z2*TH2+2.*P5*XD2*TH2*Z2/V1)+TF*(-P4*CGCF+CGCF*X2-AS*CGCF*
 3HGK/BS-P5*HGK*Z2-2.*P5*CGCF*HGK*TH2-AS*CGCF*Z2/BS-P5*Z2**2-2.*P5
 4*CGCF*TH2*Z2-GM*Z2-2.*GM*CGCF*TH2)+DH*(-P5*HGK*ZD2-P5*ZD2*Z2)+ZM*
 5(-P5*HGK*ZDD2-P5*ZDD2*Z2)-DIS*GM
  D2 = TB*(P4*XD2*TH2/V1+TH2*X2-2.*XD2*TH2*X2/V1+AS*HGK*XD2*TH2/(BS*
 1V1)-P5*HGK*TH2**2+2.*P5*HGK*XD2*TH2**2/V1-AS*TH2*Z2/BS+2.*AS*XD2
 2*TH2*Z2/(BS*V1)-2.*P5*TH2**2*Z2+3.*P5*XD2*Z2*TH2**2/V1-GKD)+TF*(
 3Z2*X2+CGCF*TH2*X2-P5*HGK*Z2*TH2-P5*CGCF*HGK*TH2**2-AS*Z2**2/BS-AS  (B100)
 4*CGCF*TH2*Z2/BS-2.*P5*TH2*Z2**2-2.*P5*CGCF*Z2*TH2**2-GM*TH2*Z2-GM*
 5CGCF*TH2**2)+DH*(ZD2*X2-P5*HGK*ZD2*TH2-AS*ZD2*Z2/BS-2.*P5*ZD2*TH2*
 6Z2)+ZM*(ZDD2*X2-P5*HGK*ZDD2*TH2-AS*ZDD2*Z2/BS-2.*P5*ZDD2*TH2*Z2)
  A31 = 0.0
  B31 = 0.0
  C31 = 1.0
  A32 = 0.0
  B32 = 0.0                                                         (B103)
  C32 = 1./TABA-TH2/SIBA**2
  A33 = 0.0
  B33 = 0.0
  C33 = -GAX2/TABA+GAX2*TH2/SIBA**2+GAZ2
  D3 = X2-GAX2*TH2/TABA+GAX2*TH2**2/SIBA**2+Z2/TABA-Z2*TH2/SIBA**2
 1+GAZ2*TH2
  PRINT 17,AS,BS,P4,P5,A11,B11,C11,A12,B12,C12,A13,B13,C13,D1,
 1A21,B21,C21,A22,B22,C22,A23,B23,C23,D2,A31,B31,C31,A32,B32,C32,
 2A33,B33,C33,D3
17 FORMAT (4E14.6//3E14.6/3E14.6/3E14.6/E14.6//3E14.6/3E14.6/3E14.6/
 1E14.6//3E14.6/3E14.6/3E14.6/E14.6//)
  D11 = A11*XD2+B11*X2+A12*ZD2+B12*Z2
  D12 = A11*X2+A12*Z2
  D13 = D1                                                          (B104)
  D21 = B21*X2+A22*ZD2+B22*Z2+A23*THD2+B23*TH2
  D22 = A22*Z2+A23*TH2
  D23 = D2
  D33 = D3
  DD4 = A11*A22*C33+A12*A23*C31-A11*A23*C32
  DD3 = A11*B22*C33+B11*A22*C33+A12*B23*C31+B12*A23*C31-A11*B23*C32
 1-B11*A23*C32-A12*B21*C33                                           (B109)
  DD2 = A11*C22*C33+B11*B22*C33+A12*C23*C31+B12*B23*C31+C12*A23*C31
 1-A11*C23*C32-B11*B23*C32-A12*C21*C33-B12*B21*C33-C13*A22*C31
  DD1 = B11*C22*C33+B12*C23*C31+C12*B23*C31+C13*B21*C32-B11*C23*C32
 1-B12*C21*C33-C12*B21*C33-C13*B22*C31
  DD0 = C12*C23*C31+C13*C21*C32-C12*C21*C33-C13*C22*C31
  U13 = D12*A22*C33+A12*A23*D33-D12*A23*C32-A12*D22*C33
  U12 = D11*A22*C33+D12*B22*C33+A12*B23*D33+B12*A23*D33-D11*A23*C32
 1-D12*B23*C32-A12*D21*C33-B12*D22*C33
  U11 = D11*B22*C33+D12*C22*C33+D13*A22*C33+A12*C23*D33+B12*B23*D33  (B108)
 1+C12*A23*D33+C13*D22*C32-D11*B23*C32-D12*C23*C32-D13*A23*C32
 2-A12*A23*C33-B12*D21*C33-C12*D22*C33-C13*A22*D33
  U10 = D11*C22*C33+D13*B22*C33+B12*C23*D33+C12*B23*D33+C13*D21*C32
 1-D11*C23*C32-D13*B23*C32-B12*D23*C33-C12*D21*C33-C13*B22*D33
  U09 = D13*C22*C33+C12*C23*D33+C13*D23*C32-D13*C23*C32-C12*D23*C33
 1-C13*C22*D33
```

```
 U23 = A11*D22*C33+D12*A23*C31-A11*A23*D33
 U22 = A11*D21*C33+B11*D22*C33+D11*A23*C31+D12*B23*C31-A11*B23*D33
1-B11*A23*D33-D12*B21*C33                                           (B152)
 U21 = A11*D23*C33+B11*D21*C33+D11*B23*C31+D12*C23*C31+D13*A23*C31
1-A11*C23*D33-B11*B23*D33-D11*B21*C33-D12*C21*C33-C13*D22*C31
 U20 = B11*D23*C33+D11*C23*C31+D13*B23*C31+C13*B21*D33-B11*C23*D33
1-D11*C21*C33-D13*B21*C33-C13*D21*C31
 U19 = D13*C23*C31+C13*C21*D33-D13*C21*C33-C13*D23*C31
 U33 = A11*A22*D33+A12*D22*C31-A11*D22*C32-D12*A22*C31
 U32 = A11*B22*D33+B11*A22*D33+A12*D21*C31+B12*D22*C31+D12*B21*C32
1-A11*D21*C32-B11*D22*C32-A12*B21*D33-D11*A22*C31-D12*B22*C31
 U31 = A11*C22*D33+B11*B22*D33+A12*D23*C31+B12*D21*C31+C12*D22*C31  (B53)
1+D11*B21*C32+D12*C21*C32-A11*D23*C32-B11*D21*C32-A12*C21*D33
2-B12*B21*D33-D11*B22*C31-D12*C22*C31-D13*A22*C31
 U30 = B11*C22*D33+B12*D23*C31+C12*D21*C31+D11*C21*C32+D13*B21*C32
1-B11*D23*C32-B12*C21*D33-C12*B21*D33-D11*C22*C31-D13*B22*C31
 U29 = C12*D23*C31+D13*C21*C32-C12*C21*D33-D13*C22*C31
 PRINT 13,D11,D12,D13,D21,D22,D23,D33,DD4,DD3,DD2,DD1,DD0,U13,U12,
1U11,U10,U09,U23,U22,U21,U20,U19,U33,U32,U31,U30,U29
 WB4 = DD3/DD4
 WB3 = DD2/DD4
 WB2 = DD1/DD4                                        (B112)
 WB1 = DD0/DD4
 W6 = -64.*WB4**6
 W5 = 96.*WB4**6
 W4 = (WB4**4)*(-48.*WB4**2-32.*WB3)                  (B122)
 W3 = (WB4**3)*(32.*WB3*WB4+8.*WB4**3)
 W2 = (WB4**2)*(16.*WB1-4.*WB3**2-4.*WB2*WB4-8.*WB3*WB4**2)
 W1 = (WB4)*(-8.*WB1*WB4+2.*WB3**2*WB4+2.*WB2*WB4**2)
 W0 = WB1*WB4**2-WB2*WB3*WB4+WB2**2
 C = 0
19 CL = C
 TOTL = TOT
 C = C+0.001
 TOT = W6*C**6+W5*C**5+W4*C**4+W3*C**3+W2*C**2+W1*C+W0   (B123)
 PRINT 13, C,TOT
 IF (TOT) 19,20,20
20 C = CL-TOTL*0.001/(TOT-TOTL)
 PRINT 13, C
 AL3 = C*WB4                                             (B124)
 DISC3 = (-WB2-3.*WB4*(AL3**2)+4.*(AL3**3)+2.*WB3*AL3)/(4.*AL3-WB4)(B125)
 IF (DISC3) 21,22,22
21 PRINT 13, DISC3                                       (B125)
 GO TO 36
22 BE3 = SQRTF (DISC3)
 AL4 = (1.-2.*C)*WB4/2.
 DISC4 = WB3-(BE3**2)-(AL3**2)-4.*AL3*AL4-(AL4**2)       (B126)
 IF (DISC4) 23,24,24
23 PRINT 13, DISC4                                       (B127)
 GO TO 36
24 BE4 = SQRTF (DISC4)
 G3 = AL3**2+BE3**2
 G4 = AL4**2+BE4**2                                      (B134)
 PRINT 13, AL3,BE3,AL4,BE4,G3,G4
 A4X = U13/DD4                                           (B111)
```

```
 A3X = U12/DD4
 A2X = U11/DD4
 A1X = U10/DD4                                              (B111)
 A0X = U09/DD4
 PRINT 13, A4X,A3X,A2X,A1X,A0X
 C1X = A0X/(G3*G4)                                          (B136)
 D1X = A4X-C1X                                              (B137)
 D2X = A3X-C1X*WB4                                          (B138)
 D3X = A2X-C1X*WB3                                          (B139)
 D4X = A1X-C1X*WB2                                          (B140)
 C6X = (2.*(AL3-AL4)*(G4*D3X-D1X*G4**2-2.*AL4*D4X)+(G3-G4)*(-G4*D2X
1+2.*AL4*G4*D1X+D4X))/(2.*(AL3-AL4)*(2.*AL3*G4-2.*AL4*G3)+(G3-G4)     (B141)
2**2)
 C5X = (D4X-G3*C6X)/G4                                      (B142)
 C4X = (G4*(D2X-2.*AL4*D1X)-D4X+C6X*(G3-G4))/(2.*G4*(AL3-AL4))   (B143)
 C3X = D1X-C4X                                              (B144)
 P23X = BE3*C3X
 P13X = C5X-AL3*C3X
 P24X =  BE4*C4X                                            (B148)
 P14X = C6X-AL4*C4X
 PRINT 13, C1X,D1X,D2X,D3X,D4X,C6X,C5X,C4X,C3X,P23X,P13X,P24X,P14X
 A4Z = U23/DD4
 A3Z = U22/DD4                                              (B154)
 A2Z = U21/DD4
 A1Z = U20/DD4
 A0Z = U19/DD4
 PRINT 13,A4Z,A3Z,A2Z,A1Z,A0Z
 C1Z = A0Z/(G3*G4)                                          (B156)
 D1Z = A4Z-C1Z
 D2Z = A3Z-C1Z*WB4                                          (B157)
 D3Z = A2Z-C1Z*WB3
 D4Z = A1Z-C1Z*WB2
 C6Z = (2.*(AL3-AL4)*(G4*D3Z-D1Z*G4**2-2.*AL4*D4Z)+(G3-G4)*(-G4*D2Z
1+2.*AL4*G4*D1Z+D4Z))/(2.*(AL3-AL4)*(2.*AL3*G4-2.*AL4*G3)+(G3-G4)
2**2)                                                                 (B163)
 C5Z = (D4Z-G3*C6Z)/G4                                      (B159)
 C4Z =   (G4*(D2Z-2.*AL4*D1Z)-D4Z+C6Z*(G3-G4))/(2.*G4*(AL3-AL4))   (B160)
 C3Z = D1Z-C4Z                                                     (B161)
 P23Z = BE3*C3Z
 P13Z = C5Z-AL3*C3Z
 P24Z = BE4*C4Z                                             (B162)
 P14Z = C6Z-AL4*C4Z
 PRINT 13, C1Z,D1Z,D2Z,D3Z,D4Z,C6Z,C5Z,C4Z,C3Z,P23Z,P13Z,P24Z,P14Z
 A4T = U33/DD4
 A3T = U32/DD4                                              (B168)
 A2T = U31/DD4
 A1T = U30/DD4
 A0T = U29/DD4
 PRINT 13, A4T,A3T,A2T,A1T,A0T
 C1T = A0T/(G3*G4)                                          (B170)
 D1T = A4T-C1T
 D2T = A3T-C1T*WB4                                          (B171)
 D3T = A2T-C1T*WB3
 D4T = A1T-C1T*WB2
 C6T = (2.*(AL3-AL4)*(G4*D3T-D1T*G4**2-2.*AL4*D4T)+(G3-G4)*(-G4*D2T (B172)
```

```
     1+2.*AL4*G4*D1T+D4T))/(2.*(AL3-AL4)*(2.*AL3*G4-2.*AL4*G3)+(G3-G4)
     2**2)                                                          (B172)
      C5T = (D4T-G3*C6T)/G4                                  (B173)
      C4T = (G4*(D2T-2.*AL4*D1T)-D4T+C6T*(G3-G4))/(2.*G4*(AL3-AL4))  (B174)
      C3T = D1T-C4T                                          (B175)
      P23T = BE3*C3T
      P13T = C5T-AL3*C3T                                     }
      P24T = BE4*C4T                                         } (B176)
      P14T = C6T-AL4*C4T                                     }
      PRINT 13, C1T,D1T,D2T,D3T,D4T,C6T,C5T,C4T,C3T,P23T,P13T,P24T,P14T
      T = -0.100
   25 T = T+0.100
   27 EAL3T = EXPF (AL3*T)
      COB3T = COSF (BE3*T)
      SIB3T = SINF (BE3*T)
      EAL4T = EXPF (AL4*T)
      COB4T = COSF (BE4*T)
      SIB4T = SINF (BE4*T)
      X = C1X+(1./(BE3*EAL3T))*(P23X*COB3T+P13X*SIB3T)+(1./(BE4*EAL4T))* (B149)
     1(P24X*COB4T+P14X*SIB4T)
      XD = (-AL3/(BE3*EAL3T))*(P23X*COB3T+P13X*SIB3T)+(1./EAL3T)*(-P23X  (B150)
     1*SIB3T+P13X*COB3T)-(AL4/(BE4*EAL4T)'*(P24X*COB4T+P14X*SIB4T)
     2+(1./EAL4T)*(-P24X*SIB4T+P14X*COB4T;
      XDD = ((AL3**2-BE3**2)/(BE3*EAL3T))*(P23X*COB3T+P13X*SIB3T)-(2.*  (B151)
     1AL3/EAL3T)*(-P23X*SIB3T+P13X*COB3T)+((AL4**2-BE4**2)/(BE4*EAL4T))*
     2(P24X*COB4T+P14X*SIB4T)-(2.*AL4/EAL4T)*(-P24X*SIB4T+P14X*COB4T)
      Z = C1Z+(1./(BE3*EAL3T))*(P23Z*COB3T+P13Z*SIB3T)+(1./(BE4*EAL4T))* (B163)
     1(P24Z*COB4T+P14Z*SIB4T)
      ZD = (-AL3/(BE3*EAL3T))*(P23Z*COB3T+P13Z*SIB3T)+(1./EAL3T)*(-P23Z  (B164)
     1*SIB3T+P13Z*COB3T)-(AL4/(BE4*EAL4T))*(P24Z*COB4T+P14Z*SIB4T)
     2+(1./EAL4T)*(-P24Z*SIB4T+P14Z*COB4T)
      ZDD = ((AL3**2-BE3**2)/(BE3*EAL3T))*(P23Z*COB3T+P13Z*SIB3T)-(2.*  (B165)
     1AL3/EAL3T)*(-P23Z*SIB3T+P13Z*COB3T)+((AL4**2-BE4**2)/(BE4*EAL4T))*
     2(P24Z*COB4T+P14Z*SIB4T)-(2.*AL4/EAL4T)*(-P24Z*SIB4T+P14Z*COB4T)
      TH= C1T+(1./(BE3*EAL3T))*(P23T*COB3T+P13T*SIB3T)+(1./(BE4*EAL4T))* (B177)
     1(P24T*COB4T+P14T*SIB4T)
      THD= (-AL3/(BE3*EAL3T))*(P23T*COB3T+P13T*SIB3T)+(1./EAL3T)*(-P23T  (B178)
     1*SIB3T+P13T*COB3T)-(AL4/(BE4*EAL4T))*(P24T*COB4T+P14T*SIB4T)
     2+(1./EAL4T)*(-P24T*SIB4T+P14T*COB4T)
      THDD= ((AL3**2-BE3**2)/(BE3*EAL3T))*(P23T*COB3T+P13T*SIB3T)-(2.*  (B179)
     1AL3/EAL3T)*(-P23T*SIB3T+P13T*COB3T)+((AL4**2-BE4**2)/(BE4*EAL4T))*
     2(P24T*COB4T+P14T*SIB4T)-(2.*AL4/EAL4T)*(-P24T*SIB4T+P14T*COB4T)
      FBZS = -TB*TH+TB*XD*TH/V1-TF*Z-TF*CGCF*TH-DH*ZD-XM*ZDD          (B92)
      WRAT = FBZS/(V1*DIS)                                   (B214)
      VAX = XD-(HGK+Z)*THD                                   (B163)
      TT = T+T2
      PRINT 26, TT,T,X,XD,XDD,Z,ZD,ZDD,TH,THD,THDD,FBZS,WRAT,VAX
   26 FORMAT (14H   TOTAL TIME=,F11.5,5H   T=,F11.5/5H   X=,F11.5,
     16H   XD=,F11.5,7H   XDD=,F11.5/5H   Z=,F11.5,6H   ZD=,F11.5,
     27H   ZDD=,F11.5/6H   TH=,F11.5,7H   THD=,F11.5,8H   THDD=,F11.5/
     38H   FBZS=,E12.5,8H   WRAT=,F10.6,7H   VAX=,F11.5//)
      TEST1 = A11*XDD+B11*XD+A12*ZDD+B12*ZD+C12*Z+C13*TH-D13         (B95)
      TEST2 = B21*XD+C21*X+A22*ZDD+B22*ZD+C22*Z+A23*THDD+B23*THD+C23*TH-  (B100)
     1D23
      TEST3 = C31*X+C32*Z+C33*TH-D33                                 (B103)
```

```
      PRINT 13, TEST1,TEST2,TEST3
      IF (VAX) 30,30,31
   30 IF (VAX+0.02) 29,28,28
   29 T = T-0.005
      GO TO 27
   31 IF (VAX-0.02) 28,28,25
   28 TT3 = TT
      T3 = T
      X3 = X
      XD3 = XD
      XDD3 = XDD
      Z3 = Z
      ZD3 = ZD
      ZDD3 = ZDD
      TH3 = TH
      THD3 = THD
      THDD3 = THDD
      FBZ3 = FBZS
      WRAT3 = WRAT
      VAX3 = VAX
      PRINT 32, TT3,T3,X3,XD3,XDD3,Z3,ZD3,ZDD3,TH3,THD3,THDD3,FBZ3,
     1WRAT3,VAX3
   32 FORMAT (17H   STATE 3 VALUES/14H   TOTAL TIME=,F11.5,6H   T3=,
     1F11.5/6H   X3=,F11.5,7H   XD3=,F11.5,8H   XDD3=,F11.5/6H   Z3=,
     2F11.5,7H   ZD3=,F11.5,8H   ZDD3=,F11.5/7H   TH3=,F11.5,8H   THD3=,
     3F11.5,9H   THDD3=,F11.5/8H   FBZ3=,E12.5,9H   WRAT3=,F10.6,
     48H   VAX3=,F11.5//)
      PRAT = FZC2/FBZ3
      IF (PRAT-1.0) 42,44,44
   42 PRINT 43, PRAT
   43 FORMAT (46H   CAUTION, CRUSHING FORCE / SLIDING FORCE IS ,F8.5//)
   44 GAX3 = P4-X3                                                (B87)
      GAZ3 = HGK+Z3                                               (B86)
      Q1 = CGCF+GAX3                                              (B192)
      A4 = GAZ3/(Q1*TF)-GM*CGCF/(TF*Q1**2)-GM*GAX3/(TF*Q1**2)
      B4 = GAX3+GAZ3*TH3+GAZ3*Z3/Q1-GAZ3*GAX3*TH3/Q1+DIS*GM/(Q1*TF)
     1+GM*Z3/Q1-GM*GAX3*TH3/Q1-2.*GM*CGCF*Z3/Q1**2+2.*GM*CGCF*GAX3*TH3/
     2Q1**2-2.*GM*GAX3*Z3/Q1**2+2.*GM*GAX3**2*TH3/Q1**2          (B194)
      C4 = DIS*GM*Z3/Q1-DIS*GM*GAX3*TH3/Q1+TF*GM*Z3**2/Q1-TF*GM*Z3*GAX3*
     1TH3/Q1-TF*GM*GAX3*TH3*Z3/Q1+TF*GM*GAX3**2*TH3**2/Q1-(TF*GM/Q1**2)*
     2(CGCF*Z3**2+CGCF*GAX3**2*TH3**2-2.*CGCF*GAX3*Z3*TH3+GAX3*Z3**2+
     3GAX3**3*TH3**2-2.*GAX3**2*Z3*TH3)
      PRINT 13,GAX3,GAZ3,Q1,A4,B4,C4
      DISC5 = (B4**2)-4.*A4*C4
      IF (DISC5) 34,33,33
   34 PRINT 13, DISC5
      GO TO 36                                                    (B195)
   33 RAD = SQRTF (DISC5)
      FBZ4 = (-B4+RAD)/(2.*A4)
      WRAT4 = FBZ4/(V1*DIS)                                       (B214)
      TH4 = -FBZ4/(Q1*TF)-Z3/Q1+GAX3*TH3/Q1                       (B193)
      Z4 = Z3+GAX3*(TH4-TH3)                                      (B186)
      X4 = X3
      PRINT 35, X4,Z4,TH4,FBZ4,WRAT4
   35 FORMAT (17H   STATE 4 VALUES/6H   X4=,F11.5,6H   Z4=,F11.5,
```

```
     17H   TH4=,F11.5/26H   VERTICAL FORCE AT BOW =,E12.5/16H   WHITE RA
     2TIO =,F10.6//)
      COBA5 = COSF(BA+TH4)
      SIBA5 = SINF(BA+TH4)
      A7 = COSA*COBA5+FS*SIBA5                                  (B200)
      B7 = -COSA*SIBA5+FS*COBA5                                 (B201)
      ET = FBZ4/((A7/B7)*COSF(TH4)-SINF(TH4))                   (B202)
      RAT = ET/TB
      PRINT 40,ET,RAT
   40 FORMAT (22H   EXTRACTING THRUST =,E12.5/50H   RATIO OF EXTRACTING
     1THRUST TO BOLLARD THRUST IS ,F15.3//)
      GO TO 36
      END
      DATA
        250.000          62.000          25.750   11880000.000
          0.523           0.886          10.000          0.724
         -4.300          -3.300          22.750          6.750
      50000.000         240.000           0.200          0.800
      43200.000
        250.000          62.000          25.750   11880000.000
          0.523           0.886          10.000          0.724
         -4.300          -3.300          22.750          6.750
      50000.000         240.000           0.200          0.800
      50000.000
        250.000          62.000          25.750   11880000.000
          0.523           0.886           6.000          0.724
         -4.300          -3.300          22.750          6.750
      50000.000         240.000           0.200          0.800
      43200.000
        250.000          62.000          25.750   11880000.000
          0.523           0.886           6.000          0.724
         -4.300          -3.300          22.750          6.750
      50000.000         240.000           0.200          0.800
      50000.000
```

The most important output of the program is the relatively sustained downward force under the bow during State 4.

F_{BZ4} = Vertical Force at Bow, lbs.

In addition other output is available as follows:

X4 = Forward motion from initial point of contact, ft.

Z4 = Vertical position of the center of gravity relative to the original position at the time of contact, ft.

TH4 = θ_4, Final trim, radians

WRAT = "White Ratio" = $\frac{F_{BZ4}}{(\text{Displacement})(\text{Impact velocity})}$ sec/ft

ET = Extracting thrust, lbs.

RAT = Extracting thrust/Bollard thrust, dimensionless.

Other information is readily available (if desired) as a function of time.

Forward position and its derivatives X, XD, XDD = x, $\dot{x}$, $\ddot{x}$ (ft, ft/sec, and ft/sec^2)

Vertical position of the center of Z, ZD, ZDD = z, $\dot{z}$, $\ddot{z}$ gravity and its derivatives (ft, ft/sec, and ft/sec^2)

TH, THD, THDD = θ, $\dot{\theta}$, $\ddot{\theta}$ Pitch angle and its derivatives (radians, rad/sec, and rad/sec^2)

F_{BZ} Downward force under bow during all phases as a function of time. lbs.

Other output is available directly but is only incidental to the solution of the basic problem. This includes total mass, including virtual (in each sense, x, z, θ), radius of gyration, pounds per foot immersion, pitch damping coefficient, heave damping coefficient, and scores of coefficients used in the solution.

Suitable Simplifications

From Milano's work (18) it may be seen that the longitudinal inertia coefficient of the waterplane is approximately linear as a function of α, the waterplane coefficient.

This may be expressed as

$$C_{i\ell} = 0.030 + \frac{(0.060 - 0.030)(\alpha - 0.65)}{(0.88 - 0.65)}$$

$$C_{i\ell} = 0.030 + \frac{0.030}{0.23} \alpha$$

$$C_{i\ell} = 0.030 + 0.1304 (\alpha - 0.65) \qquad (B203)$$

$$BM_L = \frac{I}{W_{vol}} = \frac{C_{i\ell} BL^3}{35 (DIS)}$$

$$BM_L = \frac{I}{vol} = \frac{C_{i\ell} BL^3}{(DIS)/64.2} = \frac{64.2 C_{i\ell} BL^3}{DIS} \qquad (B204)$$

$$C_b = \frac{DIS}{64.2 LBH} \qquad (B205)$$

Then,

$$BM_L = \frac{C_{i\ell} L^2}{C_b H} \qquad (B206)$$

$$KB = \frac{\alpha}{C_b + \alpha} H \qquad (B207)$$

It is noted from the "Wind Class" inclining experiment (29) that the height of the center of gravity above the keel is 23.4 ft at a draft of 26.25 ft, or

$$KG = 0.89\ H \qquad \text{(B208)}$$

Milano indicates the center of gravity may be expressed as follows:

$$0.95\ H = KG = 1.20\ H \qquad \text{(B209)}$$

Let it be assumed, as an approximation that

$$KG = 0.95\ H \qquad \text{(B210)}$$

The longitudinal metacentric height, GM_L, may be determined by using equations (B205), (B203), (B207), (B206, and (B210).

$$GM_L = KG + BM_L - KG \qquad \text{(B211)}$$

Bollard thrust may be approximated by using a propeller loading factor, $T_B/(\text{Prop. diam.})^2$

It has been shown (for twin screw icebreakers) that the ratio of propeller diameter to draft varies linearly with design draft (18).

$$PD/_H = 0.82 - \frac{(0.22)}{(22)}(H-10) = 0.82 - \frac{(H-10)}{100}$$

It follows that

$$\text{Prop. Diam.} = H\left[0.82 - \frac{1}{100}(H - 10)\right] \qquad \text{(B212)}$$

For twin screw icebreakers the propeller diameter can be based on equation (B212). From this the bollard thrust can be approximated (for icebreakers over 300 ft) by (18),

$$\overset{*}{T}_B = (0.38)(2240)(\text{Prop Diam})^2 \qquad \text{(B213)}$$

Parametric Study, General

It is important to determine how the variation of a parameter effects the sustained downward force. For example, it would seem obvious that an increase in displacement would yield a greater downward force. It is to be noted, however, that there are sixteen variables as parameters and only a few of these can be considered as approximately independent (i.e., the bow angle).

As a first basis, assume an icebreaker the size of a "Wind Class" and see what effect there would be from shifting some parameters independently within reasonable limits.

"Wind Class" Parameters

BP (length between perpendiculars) = 250.0 ft.
B (beam at waterline) = 64.0 ft.
H (draft mean) = 25.75 ft.
DIS (displacement) = 12,100,000.0 lb.
BA (i_B, bow angle) = 0.523 rad.
SA (β, spread angle complement) = 0.886 rad.
VI (impact velocity) varies from 0 to 25.0 ft/sec.

AL (α, waterplane coeff.) = 0.724

CF (LCF, center of flotation) = -1.25 ft

CG (LCG, center of gravity) = -2.40 ft

GK (KG, height of center of gravity) = 23.4 ft.

D (height of thrust line at c.g.) = 16.0 ft.

TB (bollard thrust) = 270,000 max.

GM (GM_L, long. metacentric height) = 195.6 ft.

FK (ice/ship kinetic friction) = 0.2

FS[*] (ice/ship static friction) = 0.8

SIG (failure stress of ice) = 144,000.0 lb/ft^2

Some of these properties may be varied independently (i.e. Vl, FK, SIG). Other parameters may be varied within reasonable limits and under that condition it may be assumed that they are independent (i.e., TB depends on shaft r.p.m., used during the crushing and sliding, D, GK, CG, SA, BA). The remaining parameters (GM, CF, AL, DIS, H, B, BP) may not be varied independently.

The impact velocity, Vl, will be varied on subsequent solutions (from 0 to 25.0 ft/sec, 0 to 14.8 knots) along with one other parameter.

The ice/ship kinetic friction, FK, will be varied from 0.1 to 0.3. These are reasonable limits (14), (15).

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[*] FS is not necessary for the solution of the downward force.

The compressive failure stress of the ice, SIG, will be varied from 25,000 lb/ft^2 to 200,000 lb/ft^2. (30) (Strengths below that would probably not give a "Wind Class" icebreaker any difficulty at all (37).) It should be noted this parameter cannot be controlled.

As the solution was worked out, it was presumed the bollard thrust (TB) would be based on the rpm of the shaft necessary to maintain impact velocity in open water. The variation in practice, depending on the choice of the Commanding Officer, is from zero-thrust (stopping ships screws at the time of initial contact) to maximum thrust (by applying full power at the time of initial contact, as was done during the 1963 D.T.M.B. - Westwind tests (37)). In any event, the solution considers that only a partial thrust is used against the ice until the ship stops. At that point only, bollard thrust is completely against the ice. For that reason, the bollard thrust may be considered independent and will be varied from 0 to 270,000 lbs. for the "Wind" class, the maximum available. (Other classes will have different limits.)

The height of the thrust line of action, D, measured near the center of gravity could reasonably be varied from 10.0 ft to 18.0 feet for the "Wind" class. It is noted that this is merely an extrapolation of the shafting line and may not in fact truly represent the line of action of the thrust. The solution disregards any vertical component of thrust when the ship is in trim. The solution does take into account a vertical component as the ship has it's bow raised by the ice.

The height of the center of gravity above the keel, GK, is 0.95 to 1.2 times the mean draft for most icebreakers. In normal load the "Wind Class" GK equals 0.91 times the mean draft. Therefore, GK will be varied from 22.0 ft. to 31.0 ft. for the "Wind" class. It must be noted that GM_L must be varied accordingly to keep GM_L constant.

The longitudinal position of the center of gravity, CG, may be varied but this implies there is an initial trim which effects, among other things, the effective bow angle. The secondary effects will be disregarded. (For example, if the icebreaker is up 30" by the bow as the result of shifting the center of gravity back two feet, the effective bow angle is increased about 2 o/o. CG will be varied from -4.4 ft to -1.4 ft. for the "Wind" Class.)

The spread angle complement (the angle from a normal to the hull plating to the center line plane, β), SA, may be considered as quite independent. A "sharp" bow may have $\beta = 70^\circ$ while a "blunt" bow may have $\beta = 20^\circ$. Therefore, SA will be varied from 1.2 radians (sharp) to 0.35 radians (blunt).

Probably the most often discussed variable of icebreaker design is the bow angle, BA, (i_B, the angle from the base line to the stem). Assuming that the stem is a straight line from the forward perpendicular back down to the keel, as this solution does, the lower limit must be of the magnitude of 15°. (At about 6° the stem becomes the keel of an icebreaker with a large designed drag!) In fact, at this low angle the bow

angle cannot be considered completely independent. However, the bow angle will be varied from 0.262 radians (15°) to 0.80 (about 45°).

"White Ratio"

It is anticipated that the downward force under the bow in the static condition (State 4) following ramming will be effected approximately linearly by displacement and impact velocity. For that reason, the following coefficient may be of use in comparison of parameter effects.

$$\mathrm{WRAT} = \frac{F_{BZ4}}{(\mathrm{DIS})(\mathrm{V1})} \qquad \mathrm{sec/ft} \qquad (\mathrm{B214})$$

"Glacier" Class Parameters

BP = 290.0 ft

B = 72.5 ft

H = 28.0 ft

DIS = 8640 tons = 19,350,000.0 lb

BA = 30° = 0.523 radians

SA = 50.8° = 0.886 radians

V1 = variable

AL = 0.8

CF = -1.45 ft (scaled from Wind Class length)

CG = -2.78 ft (scaled from Wind Class length0

GK = 24.5 ft (scaled from Wind Class draft)

D = 16.8 ft (scaled from Wind Class draft)

TB = 455,000.0 lb (4)

GM_L = 275.0 ft

FK = 0.2

FS = 0.8

SIG = 144,000 lb/ft^2 (30)

"Lenin" Class Parameters (4)

BP = 420.0 ft

B = 90.0 ft

H = 30.25 ft

DIS = 16,000 tons = 35,800,000.0 lbs.

BA = 30° = 0.523 radians

SA = 50.8° = 0.886 (est. equal to Glacier, Wind Class)

Vl = variable

AL = 0.80

CF = -2.10 (scaled from Wind Class length)

CG = -4.04 (scaled from Wind Class length)

GK = 27.5 (scaled from Wind Class draft)

D = 18.8 (scaled from Wind Class draft)

TB = 730,000.0 lb

GM_L = 545.0

FK = 0.2

FS = 0.8

SIG = 144,000 lb/ft^2 (30)

As noted previously, some of the variables may not be varied independently. For example, a change in α, the waterplane coefficient will cause a change in pounds-per-foot-immersion (TF), the height of the center of buoyancy (KB), and the distance from the center of buoyance to the longitudinal metacenter (BM_L). There could be a change in the height of the center of gravity but it will be assumed this is unchanged. For the sake of comparison it shall be assumed that the displacement does not change (the block coefficient remains constant). This implies that if the higher waterplanes have an increased coefficient, the lower waterplanes must have a decreased coefficient.

Assume that α is changed over a range from 0.70 to 0.85, then,

$$C_{i\ell} = 0.030 + 0.1304\,(\alpha - 0.65) \tag{B203}$$

$$C_b = \frac{DIS}{64.2\ LBH} \tag{B205}$$

$$GM_L = \frac{C_{i\ell}\,L^2}{C_b\,H} \tag{B206}$$

$$KB = \frac{\alpha\,H}{C_b + \alpha} \tag{B207}$$

$$GM_L = KB + BM_L - KG \tag{B211}$$

Utilizing the equations listed above the computer program may then give results based on a change of α and consequently a change of GM_L and pounds-per-foot immersion (the latter is already a calculation contained in the program).

It is possible to assume that the longitudinal position of the center of flotation varies slightly. Within reasonable limits other terms may be held constant even though a change in ship form would be necessary so as not to introduce trim. (However, displacement, length, draft, and the coefficients could remain constant.) It will be assumed that $\frac{LCF}{LBP}$ varies from 0 to -0.010. (For the Wind Class $LLF/_{LBP}$ = -0.005.)

In order to find the effect caused by changing draft, H, the beam to draft ratio may be varied ($^B/H$). Most other values will be held constant (i.e. displacement and block coefficient, length). This implies that the product B x H remains constant. Let the beam to draft ratio vary from 2.0 to 4.0.

First determine the product of the parent ship.

$$B \times H = C_{BH} \qquad (B215)$$

$$B = (B - H \text{ Ratio})\, H \qquad (B216)$$

$$H = \sqrt{\frac{C_{BH}}{(B\text{-}H \text{ ratio})}} \qquad (B217)$$

Using the value of C_{BH} determined in equation (B215), find the new draft H from equation (B217) and then find the new beam, B, from equation (B216).

Assume, for the sake of comparison, that

$$KG = 0.95\ H$$

Then utilize equations (B203), (B205), (B206, (B207), and (B211) to determine the corresponding longitudinal metacentric height, GM_L. By entering these changes into the program and by varying beam-to-draft-ratio as indicated, the corresponding effect may be obtained.

In most polar icebreakers the length-to-beam-ratio is approximately 4 to 1. It is possible to determine the effect of varying this ratio, $^{BP}/_{B}$, by modifying the solution but holding displacement, draft, and the block and waterplane coefficients constant. This implies BP x B remains constant.

$$BP \times B = C_{BPB} \tag{B218}$$

$$BP = (BP\text{-}B \text{ ratio})B \tag{B219}$$

$$B\ (BP\text{-}B \text{ ratio})B = C_{BPB} \tag{B220}$$

$$B = \sqrt{\frac{C_{BPB}}{(BP\text{-}B \text{ ratio})}} \tag{B221}$$

Using the value of C_{BPB} determined in equation (B218), find the new beam, B, from equation (B221) and then find the new length, BP, from

equation (B219).

Then utilize equations (B203), (B205), (B206), (B207), and (B211) to determine the corresponding longitudinal metacentric height, GM_L. By entering these changes into the program and by varying the length-to-beam ratio from 3.5 to 5.0, the corresponding effect may be determined.

One method of variation of displacement is to vary the block coefficient while holding length, BP, draft, H, and beam, B, constant. (Assume that the waterplane coefficient of the waterplane of the effective draft remains constant.)

$$DIS = C_b \, (69.2) \, LBH$$

Then utilize equations (B203), (B206), (B207), and (B211) to determine the corresponding longitudinal metacentric height. By entering these changes into the program and by varying the block coefficient, C_b, from 0.4 to 0.7, the corresponding effect may be determined.

The effect brought about by changing displacement may be examined by increasing the size of the ship such that the new ship is geometrically similar. To do this, multiply TB (assuming a constant thrust-to-displacement-ratio), and DIS by $(\text{Scale ratio})^3$. Multiply the following length dimensions by the (Scale ratio):

BP, B, H, CF, CG, GK, D, and GM_L.

By utilizing these changes the effect of varying the scale ratio from 0.8 to 1.6 may be determined for geometrically similar ships.

Model Parameters

Let the length of the ship divided by the length of the model equal lambda.

$$\frac{L_s}{L_m} = \lambda \qquad \text{(B222)}$$

It then follows that the linear dimensions must be multiplied by $1/\lambda$.

$$\begin{array}{lcll}
\underline{\text{Model}} & & \underline{\text{Ship}} & \\
BP_m & = & BP_s/\lambda & \text{(B223)} \\
B_m & = & B_s/\lambda & \\
H_m & = & H_s/\lambda & \\
CF_m & = & CF_s/\lambda & \\
CG_m & = & CG_s/\lambda & \\
GK_m & = & GK_s/\lambda & \\
D_m & = & D_s/\lambda & \\
GM_{Lm} & = & GM_{Ls}/\lambda &
\end{array}$$

Assuming a constant density fluid (fresh water or sea water),

$$DIS_m = DIS \;/\; \lambda^3 \qquad \text{(B224)}$$

Assuming a constant thrust-to-displacement ratio,

$$TB_m = TB_s / \lambda^3 \qquad \text{(B225)}$$

The coefficients are dimensionless and are not changed.

$$AL_m = AL_s \qquad FK_m = FK_s \qquad FS_m = FS_s \qquad \text{(B226)}$$

Angles are not changed.

$$BA_m = BA_s \qquad SA_m = SA_s \qquad \text{(B227)}$$

The value for compressive failure stress of ice (or the model-supporting-medium) must be changed.

$$\text{Pounds force} = \rho g L^3$$

where $\rho_m = \rho_s$ and $g_m = g_s$

$$SIG = \frac{\text{Pounds force}}{L^2} = \frac{\rho g L^3}{L^2} = \rho g L$$

It follows that

$$SIG_m = \rho_m g_m L_m \qquad SIG_s = \rho_s g_s L_s$$

$$\frac{SIG_m}{SIG_s} = \frac{\rho_m g_m L_m}{\rho_s g_s L_s} = \frac{L_m}{L_s} = \frac{1}{\lambda}$$

Therefore

$$SIG_m = SIG_s/\lambda \qquad (B228)$$

Since gravity and dynamics are involved it follows that

$$\frac{(Vl)_m^2}{g_m L_m} = \frac{(Vl)_s^2}{g_s L_s}$$

$$\frac{Vl_m}{\sqrt{L_m}} = \frac{Vl_s}{\sqrt{L_s}}$$

It is apparent that the respective Froude Numbers must be equal and

$$Vl_m = Vl_s/\sqrt{\lambda} \qquad (B229)$$

If these model parameters are used, the values of x_m and z_m for the model in its final position should equal x_s/λ and z_s/λ respectively. θ_m in its final position should be equal to θ_s. The values for force (both crushing peak and final sustained value) should be related by λ^3.

$$F_m = F_s/\lambda^3 \qquad (B230)$$

Equation (B38) is used in the program in place of equation (B41) because it is more suitable for both model and ship.

The relationship of time of events for the ship to the time of events for the model may be developed as follows:

Since $$\frac{V_m^2}{L_m} = \frac{V_s^2}{L_s}$$

$$\frac{L_m^2/T_m^2}{L_m} = \frac{L_s^2/T_s}{L_s}$$ where T = time

$$\frac{L_m}{T_m^2} = \frac{L_s}{T_s^2}$$

$$\frac{T_s}{T_M} = \sqrt{\frac{L_s}{L_m}} = \sqrt{\lambda} \qquad \text{(B231)}$$

C. SYMBOLS AND THEIR TITLES

This appendix list the symbols generally used in this research.* Symbols of special or limited use are defined separately as used in this research.

a - Acceleration, linear

A - Area in general

B - Beam at the designed waterline

C_B - Coefficient, block = $\nabla/(LB_XH_X)$

e - Base of Napierian or natural logarithms

e - Coefficient of restitution

f_k = Coefficient of kinetic friction of ice and hull

f_s = Coefficient of static friction of ice and hull

F = Force in general

F_X = Component of force in x direction in lb.

F_Z = Component of force in Z direction in lb.

F_B = Force against bow perpendicular to stem in ℄ plane, in lb.

F_{BZ} = Component of force against bow in Z-direction in lb.

F_{BX} = Component of force against bow in X-direction in lb.

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* These are, for the most part, in agreement with recommendations of the Society of Naval Architects and Marine Engineers

g - Acceleration due to gravity

H - Draft of a floating body or ship

i_B - Slope, bowline or buttock with reference to baseplane, stem angle.

L - Length, the principal longitudinal dimension of a ship, generally length between perpendiculars

$\otimes$ - Amidships in general

T_{IB} = Thrust, available for breaking ice, lb, (Total thrust - thrust used to overcome non-ice resistance)

t - Thrust-deduction fraction, $= (T - R_T)/T$

t - Time in general

T - Thrust; usually ahead thrust; specifically, thrust developed by a propulsion device, lb.

v - Velocity, linear

w - Wake fraction of Taylor

β = Angle with respect to the ℄ plane of a normal to the shell at the bow. (Note that this is the complement of half the angle of "spread" as one looks down the stem line), deg.

x - Longitudinal body axis, positive forward

z - Vertical body axis, positive from deck to keel

α - Designed waterplane coefficient

Δ - (delta, large capital) - Displacement weight in lbs of salt water

η - Efficiency, general

θ - Angle of pitch or trim in a ship, with reference to the designed or normal attitude in the fore-and-aft plane. Its natural tangent is the algebraic difference of the changes in elevation of the designed waterline at the end perpendiculars, divided by the length L.

λ - Ratio, linear or scale, full-size body or ship to model, generally expressed as a number greater than unity; for example, 20th scale or 1:20 model.

CB - Center of buoyance of a body or ship

CF - Center of flotation; geometric or moment center of the surface waterplane area A_W

CG - Center of gravity or center of mass of a body or ship

CM_L, M_L - Meacenter, for longitudinal inclination

BM_L - Metacentric radius for longitudinal inclination

GM_L - Metacentric height, longitudinal, from CG to CM_L, for longitudinal inclination

LCG - Longitudinal center of flotation abaft ℄

LCG - Longitudinal center of gravity abaft ℄

Abbreviations for Units of Measurement

ft - foot

ft^2 - square foot

ft^3 - cubic foot

lb - pound

lb ft^2 - pounds per square foot

ft-lb - foot-pound

hp - horsepower

ehp - effective power, in English horses

ihp - indicated power, in English horses

shp - shaft power, in English horses

sec - second

rpm - revolutions per minute

kt - knot, one nautical mile per hour

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E. SAMPLE CALCULATIONS

Sample Calculation using Runeberg's Equation

Wanted: F_{BZ} for U.S.S. Glacier

Given (14):

$$T_{IB} = 160 \text{ tons}$$

$$T_{IB} = (2240)(160) = 358{,}400 \text{ lb.}$$

$$i_B = 30.0^{\circ}$$

$$\beta = 40.3^{\circ}$$

$$f_k = 0.05$$

Solution:

$$\cos i_B = 0.866$$

$$\sin i_B = 0.500$$

$$\cos \beta = 0.763$$

$$F_{BZ} = \frac{T_{IB} (\cos i_B \cos \beta - f_k \sin i_B)}{(\sin i_B \cos \beta + f_k \cos i_B)} \qquad \text{(A12)}$$

$$F_{BZ} = \frac{T_{IB} \left[(0.866)(0.763) - (0.05)(0.500)\right]}{\left[(0.500)(0.763) + (0.05)(0.866)\right]}$$

$$F_{BZ} = T_{IB} \frac{(0.660) - (0.025)}{(0.381) + (0.043)}$$

$$F_{BZ} = T_{IB} \left(\frac{0.635}{0.424}\right) = 1.50\ T_{IB}$$

Glacier

$$F_{BZ} \doteq 537{,}600 \text{ lb}$$

Wanted: F_{BZ} for Stalin Class U.S.S.R.

Given (3): $i_B = 25.0^\circ$

i_E = Half Entrance angle in ωL plane = 21°

$f_k = 0.05$

Solution: $\tan \beta = \dfrac{\sin i_B}{\tan i_E}$

$$\tan i_E = \tan 21^\circ = 0.384$$

$$\sin i_B = \sin 25^\circ = 0.423$$

$$\tan \beta = \frac{0.423}{0.384} = 1.100$$

$$\beta = 47.8^\circ \qquad \cos \beta = 0.672$$

$$\cos i_B = \cos 25^\circ = 0.906$$

$$F_{BZ} = \frac{T_{IB}\left[(0.906)(0.672) - (0.05)(0.423)\right]}{\left[(0.423)(0.672) + (0.05)(0.906)\right]}$$

$$F_{BZ} = T_{IB} \left[\frac{(0.608) - (0.021)}{(0.284) + (0.045)} \right]$$

$$F_{BZ} = T_{IB} \left(\frac{0.587}{0.329} \right) = \boxed{1.78\ T_{IB}}$$

Stalin

Sample Calculation using Kari's Equation

Wanted: F_{BZ} for U.S.S.Glacier

Given (14):

Δ = 8640 tons

θ = Variable

L = 290 ft.

H = 28 ft.

C = 0.07

Solution:

$$F_{BZ} = \frac{4480\ \Delta\ C\ L\ \sin\theta}{H} \qquad \text{(A26)}$$

$$F_{BZ} = \frac{(4480)(8640)(0.07)(290)}{(28)}$$

$$\boxed{F_{BZ} = 28{,}100{,}000\ \sin\theta}$$

		Glacier
θ	$\sin\theta$	F_{BZ}
0	0	0
1°	0.01745	491,000 lb.
2°	0.03490	981,000
3°	0.05234	1,470,000
4°	0.06976	1,960,000
5°	0.08716	2,450,000

Wanted: F_{BZ} for Stalin Class U.S.S.R.

Given (3):

Δ = 11,000 tons

θ = Variable

L = 335 ft.

H = 29.5 ft.

C = 0.07 ft.

Solution:

$$F_{BZ} = \frac{4480\ \Delta\ C\ L\ \sin\theta}{H} \qquad \text{(A26)}$$

$$F_{BZ} = \frac{(4480)(11,000)(0.07)(335)}{(29.5)} \sin\theta$$

$$\boxed{F_{BZ} = 39,200,000\ \sin\theta}$$

		Stalin
θ	$\sin\theta$	F_{BZ}
0	0	0
1°	0.01745	684,000 lb
2°	0.03490	1,365,000
3°	0.05234	2,045,000
4°	0.06976	2,730,000
5°	0.08716	3,415,000

Sample Calculation using Simonson's Equation

Wanted: F_{BZ} for U.S.S. Glacier

Given (14): $T_{IB} = 160$ tons

$T_{IB} = 358{,}400$ lb

$i_B = 30.0^\circ$

Solution:

$$F_{BZ} = \frac{T_{IB}}{\tan(i_B + \theta)} \tag{A43}$$

$$\boxed{F_{BZ} = \frac{T_{IB}}{\tan(30^\circ + \theta)}}$$

Glacier

$$\theta = 0 \qquad F_{BZ} = \frac{T_{IB}}{0.577} = 1.73\ T_{IB}$$

$$\theta = 2^\circ \qquad F_{BZ} = \frac{T_{IB}}{0.625} = 1.60\ T_{IB}$$

Try $\theta = 0$ $T_{IB} = 358{,}400$ lb

$$F_{BZ} = (1.73)(358{,}400) = 620{,}000 \text{ lb}$$

A check against Figure II, Kari's Equation, shows that the trim would be about 1°.

Try $\theta = 1^\circ$ $T_{IB} = 358{,}400$ lb

$$F_{BZ} = (1.66)(358{,}400) = \boxed{595{,}000 \text{ lb}}$$

This is in approximate agreement with the trim indicated by Kari.

Wanted: F_{BZ} for Stalin Class U.S.S.R.

Given (3): $i_B = 25.0^\circ$

$$\boxed{F_{BZ} = \frac{T_{IB}}{\tan(25^\circ + \theta)}}$$

$\theta = 0^\circ$ $\qquad F_{BZ} = 2.14\ T_{IB}$

$\theta = 2^\circ$ $\qquad F_{BZ} = 1.96\ T_{IB}$

Sample Calculation based on This Research

Assume the following parameters are known or can be suitably approximated:

Ship

L = Length between perpendiculars, ft.

B = Waterline beam, ft.

H = Normal draft, ft.

Δ = Normal displacement, lb.

i_B = Angle from base line to stem, radians

β = Angle of normal to bow plating with respect to centerline plane, radians

v_1 = Velocity of ship immediately prior to initial contact, ft/sec.

α = Waterplane coefficient, dimensionless

LCF = Distance from amidships to center of floation (+ if forward, - if aft), ft.

LCG = Distance from amidships to center of gravity (+ if forward, - if aft), ft.

KG = Height of center of gravity above base line, ft.

d = Height of propeller hub above base line, ft.

T_{BOL} = Bollard thrust at same rpm as that needed to maintain v_1 at approach, lb.

GM_L = Longitudinal Metacentric Height, ft.

Ice

f_k = Coefficient of kinetic friction between ice and ship, dimensionless.

f_s = Coefficient of static friction between ice and ship, dimensionless.

σ = Compressive failure stress of ice (during the local crushing) lb/ft^2

Values will be selected (or assumed) based on the C.G.C. Westwind.

$L = 250.0$ ft.

$B = 62.0$ ft.

$H = 25.75$ ft.

$\Delta = (2240)(5300) = 11.88 \times 10^6$ lb.

$i_B = 30^\circ = \frac{30}{57.3} = 0.523$ radians

$\beta = 50.8^\circ = \frac{50.8}{57.3} = 0.886$ radians

$v_1 = 6.0$ ft/sec

$\alpha = 0.724$

LCF = - 4.3 ft (assumed)

LCG = -3.3 ft (assumed)

KG = 22.75 ft (assumed)

d = 6.75 ft (assumed)

T_{BOL} = 50.0×10^3 lbs (38)

(Note: 49.6 rpm at 0 towrope pull gives 6 knots 10 ft/sec

49.6 rpm at 0 ft/sec gives T_{BOL})

GM_L = 240 ft

f_k = 0.20 (assumed)

f_s = 0.80

σ = 347 psi = 50,000 lb/ft^2

See (21)

```
BP   =   250.000  B  =   62.000   H  =      25.750   DIS = 11880000.000
BA   =      .523  SA =     .886   V1=        6.000   AL =          .724
CF   =    -4.300  CG =   -3.300   GK=       22.750   D  =         6.750
TB   = 50000.000  GM =  240.000   FK=         .200   FS =          .800
SIG  =  50000.000

XM   =  .3992E 06               ZM  =  .6867E 06             (B66)(B67)
RG   =  .5500E 02               THM =  .1797E 10             (B71)(B68)
DP   =  .8265E 09                DH =  .3975E 06             (B69)(B70)
TF   =  .7205E 06                P1 =  .2109E 05             (B72)(B33)
P2   =  .2167E 05                                            (B39)
A1   =  .1797E 10                B1 =  .8265E 09             (B56)(B56)
C1   =  .2851E 10                P3 =  .2844E 07             (B56)(B51)
D1   =  .1024E 09                                            (B56)
AL1  =  .2300E 00               BE1 =  .1239E 01             (B57)(B57)
AA1  =  .3922E-01               AA2 =  .2409E -01            (B57)(B57)
A2   =  .6867E 06                B2 =  .3975E 06             (B40(B40)
C1   =  .7205E 06                D1 =  .7802E 06             (B40)(B40)
AL2  =  .2894E 00               BB1 = -.1405E 01             (B61)(B61)
BB1  =  .1405E 01               BB2 =  .1630E 01             (B61)(B61)

T    = -0.
TH   = -.00000     THD = -.00000      THDD = .00000          (B58)(B55)(B59)
 Z   =  .00000      ZD =  .00000       ZDD =-.00000          (B62)(B63)(B64)
 X   = 0.           XD = 6.00000       XDD =-0.              (B41)(B40)(B39)
FXC  =  .00000E  00 FZC=  .00000E 00                         (B47)(B48)
TAGA = -.00000      GAX= 128.30000     GAZ = 3.00000         (B65)(B20)
                              DIF =  .57655

T    =  .05000
TH   =  .00000     THD =  .00000       THDD= .00014
 Z   = -.00000      ZD = -.00005       ZDD =-.00281
 X   =  .30000      XD = 5.99992       XDD =-.00475
FXC  =  .18978E 04  FZC= .19504E 04
TAGA =  .00006      GAX= 128.29999     GAZ = 3.00000  DIF= .57649

       .10000
       .00000        .00002          .00056
      -.00001       -.00037         -.01114
       .59997       5.99937         -.01902
    .75906E 04     .78010E 04
```

* Numbers in parentheses indicate equation numbers of Appendix B.

```
  .00046          128.29988          2.99999          .57609


  .15000
  .00000             .00006           .00125
 -.00005            -.00125          -.02479
  .89991            5.99786          -.04278
.17077E 05         .17550E 05
  .00155          128.29938          2.99995          .57500


  .20000
  .00001             .00015           .00220
 -.00015            -.00294          -.04359
 1.19979            5.99493          -.07604
.30354E 05         .31196E 05
  .00366          128.29806          2.99985          .57290


  .25000
  .00002             .00029           .00340
 -.00036            -.00569          -.06734
 1.49959            5.99009          -.11880
.47419E 05         .48734E 05
  .00710          128.29530          2.99964          .56948


  .30000
  .00004             .00049           .00484
 -.00074            -.00975          -.09584
 1.79929            5.98288          -.17102
.68267E 05         .70159E 05
  .01218          128.29032          2.99926          .56442


  .35000
  .00007             .00078           .00651
 -.00136            -.01535          -.12887
 2.09887            5.97280          -.23272
.92893E 05         .95467E 05
  .01922          128.28217          2.99864          .55742

  .40000
  .00012             .00115           .00840
 -.00230            -.02271          -.16623
 2.39831            5.95938          -.30386
.12129E 06         .12465E 06
  .02852          128.26979          2.99770          .54819
```

```
  .45000
  .00018           .00162               .01050
 -.00366          -.03204              -.20768
 2.69759          5.94211              -.38442
.15345E 06       .15770E 06
  .04037        128.25193              2.99634           .53643

  .50000
  .00028           .00220               .01279
 -.00554          -.04354              -.25302
 2.99670          5.92049              -.47440
.18936E 06       .19461E 06
  .05507        128.22725              2.99446           .52186

  .55000
  .00041           .00290               .01525
 -.00806          -.05740              -.30201
 3.29561          5.89402              -.57376
.22903E 06       .23537E 06
  .07293        128.19424              2.99194           .50416

  .60000
  .00057           .00373               .01789
 -.01133          -.07380              -.35442
 3.59429          5.86214              -.68247
.27242E 06       .27997E 06
  .09428        128.15132              2.98867           .48304

  .65000
  .00078           .00469               .02067
 -.01548          -.09290              -.41001
 3.89275          5.82431              -.80052
.31954E 06       .32839E 06
  .11943        128.09677              2.98452           .45817

  .70000
  .00104           .00580               .02359
 -.02066          -.11485              -.46856
 4.190904         5.77993              -.92786
.37037E 06       .38063E 06
  .14874        128.02878              2.97934           .42920
```

```
 .75000
 .00136             .00705             .02664
-.02702            -.13980            -.52983
4.48886            5.72840           -1.06446
.42490E 06         .43667E 06
 .18261          127.94544            2.97298          .39576

 .80000
 .00175             .00846             .02979
-.03470            -.16788            -.59357
4.78648            5.66904           -1.21029
.48311E 06         .49650E 06
 .22146          127.84478            2.96530          .35743

 .85000
 .00221             .01003             .03304
-.04386            -.19919            -.65956
5.08378            5.60114           -1.36531
.54499E 06         .56009E 06
 .26578          127.72475            2.95614          .31373

 .90000
 .00276             .01177             .03636
-.05467            -.23387            -.72756
5.38074            5.52391           -1.52947
.61052E 06         .62744E 06
 .31613          127.58323            2.94533          .26410

 .95000
 .00339             .01367             .03975
-.06730            -.27198            -.79734
5.67735            5.43648           -1.70274
.67968E 06         .69852E 06
 .37319          127.41804            2.93270          .20789

1.00000
 .00413             .01574             .04318
-.08193            -.31363            -.86867
5.97359            5.33788           -1.88507
.75246E 06         .77331E 06
 .43777          127.22699            2.91807          .14429
```

```
 1.05000
  .00497             .01799              .04665
 -.09873            -.35887             -.94131
 6.26942            5.22699            -2.07641
.82884E  06        .85180E  06
  .51088          127.00782             2.90127            .07231

T=  1.10000
TH=  .00593         THD=  .02041      THDD=  .05014
 Z=  -.11788         ZD=  -.40777      ZDD=  -1.01505
 X= 6.56484          XD= 5.10254       XDD=  -2.27670
FXC= .90879E  06    FXC= .93397E  06
TAGA=.59378         GAX= 126.75827    GAZ= 2.88212       DIF= -.00930

T2=   1.09430
TH2=       .00582     THD2=          .02013        THDDD2=        .04974
Z2=     -.11569     ZD2=          -.40220        ZDD2=     -1.00665
X2=    6.53119      XD2=          5.11671        XDD2=     -2.25389
FXC2=   .89968E  06   FZC2=      .92461E  06
TAGA2=         .58433   GAX2=    126.78669       GAZ2=          2.88431    DIF2=    .00000

AS=    .489195E  00              BS= .448067E  00                 (B80)(B81)
P4=    .133318E  03              P5= .219200E  01                 (B87)(B99)

A11= -.399168E  06               B11= -.838691E  04
C11=   .000000E  00              A12=   .758452E  06
B12=   .439022E  06              C12=   .795773E  06               (B95)
A13=   .000000E  00              B13=   .000000E  00
C13=   .280060E  06              D1=   -.533230E  05

A21=   .000000E  00              B21= -.127029E  06
C21= -.238980E  06               A22= -.892475E  08                (B100)
A23= -.179685E  10               B23= -.826493E  09
C23= -.292629E  10               D2 = -.223974E  07

A31= .000000E    00              B31=   .000000E  00
C31= .100000E    01              A32=   .000000E  00               (B103)
B32= .000000E    00              C32=   .171112E  01
A33= .000000E    00              B33=   .000000E  00
C33= -.214063E  03               D3 =   .508733E  01
```

```
D11= - .2453E  07
D12= - .2695E  07
D13= - .5332E  05    (B104)
D21=   .5504E  05
D22= - .1327E  06
D23= - .2240E  07
D33=   .5087E -01

DD4= - .1022E  17
DD3= - .6601E  16
DD2= - .1421E  17      (B109)
DD1= - .2184E  16
DD0= - .2343E  16

U13= - .6672E  17
U12= - .9522E  17
U11= - .1204E  18       (B108)
U10= - .7252E  17
U09= - .1344E  17

U23=   .1182E  16
U22=   .4958E  16
U21=   .4045E  16       (B152)
U20=   .7220E  16
U19=   .1591E  15

U33= - .5946E  14
U32= - .2483E  15       (B166)
U31= - .1924E  15
U30= - .2292E  15
U29= - .5826E  13

0  = 1.0000E -03
TOT= -.4964E-01            (B123)
C=    .2000E-02
TOT= -.4869E-01            (B123)
C=    .3000E-02
TOT= -.4775E-01            (B123)
C=    .4000E-02
TOT= -.4681E-01            (B123)
      .5000E-02
     -.4588E-01
      .6000E-02
     -.4496E-01
      .7000E-02
     -.4404E-01
```

```
        .6300E-01
       -.3312E-02
        .6400E-01
       -.2750E-02
        .6500E-01
       -.2192E-02
  C=    .6600E-01
  TOT=-.1639E-02                (B123)
  C=    .6700E-01
  TOT=-.1092E-02                (B123)
  C=    .6800E-01
  TOT=-.5491E-03                (B123)
  C=    .6900E-01
  TOT=-.1134E-04                (B123)
  C=    .7000E-01
  TOT= .5215E-03                (B123)
  C=    .6902E-01
  AL3= .4460E-01                (B124)
  BE3= .4464E 00                (B125)
  AL4= .2785E 00                (B126)
  BE4= .1031E 01                (B127)
  G3=  .2013E 00                (B134)
  G4=  .1140E 01                (B134)
  A4X= .6531E 01
  A3X= .9321E 01                (B111)
  A2X= .1178E 02
  A1X= .7098E 01                (B11)
  A0X= .1315E 01
  C1X= .5736E 01                (B136)
  D1X= .7955E 00                (B137)
  D2X= .5615E 01                (B138)
  D3X= .3808E 01                (B139)
  D4X= .5872E 01                (B140)
  C6X= .3877E-02                (B141)
  C5X= .5152E 01                (B142)
  C4X=-.3342E-01                (B143)
  C3X= .8289E 00                (B144)
  P23X= .3700E 00
  P13X= .5115E 01
  P24X= -.3444E-01
  P14X=  .1318E-01              (B148)
```

```
A4Z= - .1157E 00
A3Z= - .4853E 00
A2Z= - .3959E 00
A1Z= - .7068E 00
A0Z= - .1557E-01      (154)

C1Z= - .6788E-01      (B156)
D1Z= - .4781E-01
D2Z= - .4414E 00
D3Z= - .3-15E 00
D4Z= - .6922E 00      (B157)

C6Z=   .1777E 00      (B158)
C5Z= - .6388E 00      (B159)
C4Z= - .9902E-01      (B160)
C3Z=   .5121E-01      (B161)

P23Z=  .2286E-01
P13Z= -.6411E 00      (B162)
P24Z= -.1020E 00
P14Z=  .2053E 00

A4T=   .5820E-02
A3T=   .2431E-01
A2T=   .1884E-01      (B168)
A1T=   .2243E-01
A0T=   .5703E-03

C1T=   .2486E-02      (B170)

D1T=   .3334E-02
D2T=   .2270E-01
D3T=   .1538E-01      (B171)
D4T=   .2190E-01

C6T=   .1439E-02      (B172)
C5T=   .1896E-01      (B173)
C4T= - .9476E-03      (B174)
C3T=   .4281E-02      (B175)

P23T=  .1911E-02
P13T=  .1877E-01
P24T= -.9766E-03      (B176)
P14T=  .1703E-02

      Total Time=   1.09430  T=   -0.
```

```
X=     6.53119    XD=   5.10089    XDD=    -.59423    (B149)(B150)(B151)
Z=    -.11569     ZD=   -.41053    ZDD=     .03023    (B163)(B164)(B165)
TH=    .00582     THD=   .02055    THDD=  -.00253     (B177)(B178)(B179)
FBZS=  .23022E 06  WRAT=   .003230  VAX=  5.04163    (B92) (B214) (B183)

Test 1= - .3564E-01     (B95)          These values are of the order of
Test 2= - .3187E 01     (B100)         1/100 of 1 o/o error in the solution
Test 3= - .2980E-06     (B103)         of the simultaneous differential
                                       equations for sliding.

Total Time=          1.19430    T=       .10000
X=   7.03814       XD=     5.03647       XDD=    -.69383
Z=    -.15662      ZD=     -.40814       ZDD=     .01790
TH=   .00786       THD=      .02027      THDD=   -.00310
FBZS=  .26219E  06   WRAT=    .003678     VAX=    4.97885

-.6396E-01
-.3438E 01
-.3576E-06
Total Time=         1.29430       T=      .20000
X=       7.53816      XD=      4.96220     XDD=      -.79118
Z=       -.19736      ZD=      -.40690     ZDD=      .00713
TH=       .00987      THD=      .01993      THDD=    -.00364
FBZS=    .29387E  06   WRAT=    .004123    VAX=    4.90635

-.8838E-01
-.3687E 01
-.3576E-06
Total Time=       1.39430        T=      .30000
X=       8.03026     XD=      4.87832       XDD=      -.88605
Z=       -.23803     XD=     -.40665        ZDD=      -.00193
TH=       .01185     THD=     .01954        THDD=      -.00415
FBZS=     .32525E  06   WRAT=   .004563     VAX=     4.82435

-.1299E 00
-.2937E 01
-.4172E-06
Total Time=       1.49430       T=      .40000
X=       8.51351     XD=     4.78508      XDD=     -.97824
Z=       -.27872     ZD=     -.40722     ZDD=      -.00917
TH=       .01378      THD=    .01910     THDD=      -.00464
FBZS=    .35626E  06   WRAT=   .004998     VAX=     4.73311
```

```
-.1597E  00
-.4688E  01
-.5960E- 06
 Total Time =       1.59430    T =           .50000
 X=      8.98698      XD=      4.68276       XDD=     -1.06756
 X=       -.31950     ZD=      -.40842       ZDD=      -.01453
 TH=       .01566      THD=       .01861      THDD=     -.00510
 FBZS=   .38686E 06     WRAT=      .005427     VAX=        4.63288

-.2139E  00
-.3187E  01
-.4768E- 06
 Total Time=          1.69430    T=            .60000
 X=      9.44977       XD=      4.57167        XDD=    -1.15384
 Z=       -.36042      ZD=      -.41007        ZDD=     -.01795
 TH=        .01750      THDD=       .01808      THDD=     -.00553
 FBZS=   .41700E 06     WRAT=       .005850       VAX=    4.52395

-.2529E  00
-.2687E  01
-.5960E- 06
 Total Time=          1.79430    T=            .70000
 X=      9.90103       XD=      4.45210       XDD=     -1.23688
 Z=      -.40152       ZD=      -.41195       ZDD=      -.01943
 TH=        .01928     THD=        .01751     THDD=      -.00593
 FBZS=   .44663E 06   WRAT=       .006266      VAX=       4.40662

-.2979E  00
-.4688E  01
-.6557E- 06
 Total Time=          1.89430    T=            .80000
 X=    10.33992        XD=      4.32441       XDD=     -1.31654
 Z=      -.44281       ZD=      -.41389       ZDD=      -.01895
 TH=        .02100      THD=       .01689      THDD=      -.00630
FBZS=   .47567E  06     WRAT=      .006673     VAX=        4.28121

-.3604E  00
-.5187E  01
-.5960E- 06
 Total Time=          1.99430    T=            .90000
 X=   10.76565         XD=     4.18892        XDD=     -1.39266
 Z=     -.48429        ZD=     -.41568        ZDD=      -.01657
 TH=      .02266       THD=      .01625       THDD=      -.00664
 FBZS=  .50407E  06     WRAT=     .007072      VAX=       4.14805
```

```
-.4160E  00
-.3687E  01
-.5960E- 06
  Total Time=        2.09430     T=       1.00000
X=    11.17745       XD=         4.04600     XDD=        -1.46510
Z=       -.52593     ZD=         -.41714     ZDD=         -.01232
TH=        .02425     THD=         .01557     THDD=        -.00694
FBZS=    .53176E 06     WRAT=      .007460     VAX=         4.00748

-.4692E  00
-.5187E  01
-.4172E- 06
 Total Time=         2.19430     T=       1.10000
 X=   11.57461        XD=        3.89602    XDD=        -1.53374
 Z=      -.56770      ZD=        -.41808    ZDD=         -.00630
 TH=       .02577      THD=        .01486    THDD=         -.00722
FBZS=     .55867E 06     WRAT=      .007838    VAX=          3.85988

-.5410E  00
-.6187E  01
-.7153E- 06
 Total Time=         2.29430     T=       1.20000
 X=  11.95644         XD=        3.73938     XDD=        -1.59846
 Z=     -.60953       ZD=        -.41834     ZDD=          .00141
TH=      .02722        THD=        .01412     THDD=        -.00746
FBZS=     .58473E 06    WRAT=        .008203     VAX=       3.70562

-.6050E  00
-.6187E  01
-.5960E- 06
 Total Time=         2.39430     T=       1.30000
 X=  12.32228         XD=       3.57646      XDD=        -1.65916
 Z=     -.65134       ZD=       -.41775      ZDD=         .01068
 TH=    .02859        THD=        .01337      THDD=       -.00767
FBZS= .60986E 06        WRAT=      .008556      VAX=       3.54507

-.6641E  00
-.4187E  01
-.5960E- 06
 Total Time=         2.49430     T=       1.40000
 X=  12.67153         XD=        3.40768    XDD=        -1.71576
 Z=     -.69304       ZD=        -.41616    ZDD=          .02138
 TH=      .02989        THD=        .01259    THDD=          -.00784
 FBZS=  .63400E 06     WRAT=      .008894   VAX=         3.37863
```

```
-.7231E  00
-.4187E  01
-.5960E- 06
 Total Time=          2.59430     T=        1.50000
 X=  13.00363         XD=         3.23345    XDD=         -1.76816
 Z=    -.73453        ZD=         -.41343    ZDD=           .03338
 TH=     .03111        THD=        .01180     THDD=         -.00799
 FBZS= .65707E 06      WRAT=        .009218   VAX=            3.20672


-.7988E  00
-.4187E  01
-.3576E- 06
 Total Time=          2.69430     T=        1.60000
 X=  13.31806         XD=          3.05419    XDD=         -1.81631
 Z=    -.77569        ZD=          -.40944    ZDD=           .04651
 TH=      .03225       THD=          .01099    THDD=          -.00811
 FBZS=  .67900E 06      WRAT=         .009526   VAX=           3.02974


-.8623E  00
-.5187E  01
-.5384E- 06
 Total Time=          2.79430     T=        1.70000
 X=  13.61432         XD=         2.87034    XDD=         -1.86015
 Z=    -.81638        ZD=         -.40410    ZDD=           .06061
 TH=      .03331       THD=         .01018    THDD=          -.00821
 FBZS=  .69972E 06     WRAT=        .009816   VAX=           2.84811


-.9185E  00
-.4187E  01
-.4768E- 06
 Total Time=          2.89430     T=        1.80000
 X=  13.89198         XD=         2.68231    XDD=         -1.89963
 Z=    -.85646        ZD=         -.39730    ZDD=           .07550
 TH=     .03429        THD=          .00935    THDD=          -.00827
 FBZS= .71916E 06      WRAT=         .010089   VAX=          2.66226


-.9795E  00
-.4187E 01
-.3576E- 06
 Total Time=          2.99430     T=        1.90000
 X=  14.15066         XD=         2.49056    XDD=         -1.93473
 Z=    -.89578        ZD=         -.38897    ZDD=           .09102
 TH=    .03518         THD=        .00853     THDD=        -.00831
 FBZS= .73727E 06      WRAT=        .010343   VAX=        2.47262
```

```
-.1033E  01
-.4187E  01
-.3576E- 06
 Total Time=        3.09430     T=       2.00000
 X=  14.38999        XD=        2.29551     XDD=         -1.96543
 Z=    -.93420       ZD=        -.37908     ZDD=          .10699
 TH=    .03599        THD=        .00769     THDD=         -.00833
 FBZS= .75397E 06      WRAT=       .010578    VAX=          2.27962


-.1090E  01
-.4187E  01
-.2980E- 06
 Total Time=        3.19430     T=       2.10000
 X=  14.60966        XD=        2.09762    XDD=          -1.99171
 Z=    -.97155       ZD=        -.36757    ZDD=           .12322
 TH=    .03672        THD=         .00686   THDD=           -.00832
 FBZS= .76921E 06      WRAT=        .010791   VAX=           2.08370


-.1140E  01
-.6187E  01
-.3576E- 06
 Total Time=        3.29430     T=       2.20000
 X=  14.80943        XD=        1.89732    XDD=          -2.01357
 Z=  -1.00766        ZD=        -.35443    ZDD=           .13955
 TH=    .03736       THD=        .00603     THDD=          -.00829
 FBZS= .78295E 06     WRAT=         .010984   VAX=          1.88530


-.1187E  01
-.4187E  01
-.1788E- 06
 Total Time=        3.39430     T=       2.30000
 X=  14.98906        XD=        1.69505    XDD=          -2.03103
 Z=  -1.04238        ZD=        -.33966    ZDD=           .15579
 TH=    .03792        THD=        .00520   THDD=          -.00824
 FBZS= .79512E 06      WRAT=        .011155   VAX=           1.68486


-.1246E  01
-.4187E  01
 .1192E- 06
 Total Time=        3.49430     T=       2.40000
 X=  15.14839        XD=        1.49126     XDD=         -2.04411
 Z=  -1.07554        ZD=        -.32328     ZDD=          .17179
 TH=        .03840    THD=        .00438     THDD=         -.00818
 FBZS=  .80568E 06     WRAT=       .011303     VAX=          1.48282
```

```
  -.1275E  01
-.5187E  01
  -.1192E- 06
   Total Time=        3.59430     T=       2.50000
   X=  15.28728       XD=       1.28637    XDD=        -2.05284
   Z=   -1.10698      ZD=       -.30532    ZDD=          .18737
   TH=    .03880       THD=        .00357   THDD=        -.00809
   FBZS= .81461E 06     WRAT=       .011428   VAX=       1.27962

  -.1324E  01
  -.5187E  01
   .2384E- 06
   Total Time=        3.69430     T=       2.60000
   X=  15.40564       XD=       1.08083     XDD=       -2.05725
   Z=   -1.13655      ZD=       -.28582     ZDD=         .20240
   TH=    .03912       THD=        .00276    THDD=        -.00799
   FBZS= .82186E  06    WRAT=       .011530   VAX=         1.07568

  -.1367E  01
  -.4187E  01
   .4768E- 06
   Total Time=        3.79430     T=       2.70000
   X=  15.50344       XD=        .87506     XDD=       -2.05740
   Z=  01.16410       ZD=       -.26486     ZDD=         .21673
   TH=     .03935      THD=       .00197      THDD=        -.00788
   FBZS= .82740E 06     WRAT=      .011608    VAX=           .87144

  -.1377E  01
-.6187E  01
   .5960E- 06
   Total Time=        3189430     T=       2.80000
   X=  15.58066       XD=        .66949    XDD=        -2.05334
   Z=   -1.18948      ZD=       -.24250    ZDD=          .23022
   TH=     .03951      THD=        .00119   THDD=        -.00775
   FBZS= .83123E 06     WRAT=       .011661   VAX=          .66734

  -.1403E  01
  -.2188E  01
   .8345E- 06
   Total Time=        3.99430     T=       2.90000
   X=  15.63735       XD=        .46453    XDD=        -2.04514
   Z=   -1.21255      ZD=       -.21885    ZDD=          .24276
   TH=     .03959     THD=         .00042   THDD=         -.00761
   FBZS=  .83332E 06   WRAT=        .011691   VAX=           .46378
```

```
-.1417E  01
-.5187E  01
 .8345E- 06
 Total Time=          4.09430      T=        3.00000
 X=  15.67360         XD=          .26060     XDD=          -2.03286
 Z=  -1.23320         ZD=         -.19399     ZDD=            .25425
 TH=     .03960        THD=         -.00033    THDD=           -.00746
 FBZS= .83366E 06        WRAT=     .011696       VAX=              .26119

-.1430E  01
-.6187E  01
 .1073E- 05
 Total Time=          4.19430      T=       3.10000
 X=  15.68952         XD=          .05809    XDD=           -2.01659
 Z=  -1.25131         ZD=         -.16804    ZDD=             .26458
 TH=    .03953         THD=          -.00107   THDD=             -.00731
 FBZS= .83226E 06       WRAT=          .011676  VAX=                .05997

-.1462E  01
-.4187E  01
 .1431E- 05
 Total Time=          4.29430      T=        3.20000
 X=  15.68528         XD=         -.14259    XDD=          -1.99641
 Z=  -1.26678         ZD=         -.14111    ZDD=            .27367
 TH=     .03938        THD=          -.00179   THDD=           -.00714
 FBZS= .82911E 06       WRAT=          .011632   VAX=             -.13948

-.1454E  01
-.2188E  01
 .1550E- 05
 Total Time=          4.28930      T=        3.19500
 X=  15.68597         XD=         -.13260     XDD=          -1.99751
 Z=  -1.26607         ZD=         -.14248     ZDD=            .27325
 TH=     .03939        THD=         -.00176    THDD=           -.00715
 FBZS= .82931E 06       WRAT=        .011635    VAX=             -.12955

-.1442E  01
-.4187E  01
 .1550E- 05
Total Time=           4.28430      T=        3.19000
X=   15.68661          XD=        -.12261     XDD=          -1.99860
Z=   -1.26535          ZD=        -.14384     ZDD=            .27282
TH=      .03940         THD=        -.00172    THDD=           -.00716
FBZS=  .82950E 06        WRAT=       .011637     VAX=             -.11963
```

```
-.1451E  01
-.4187E  01
 .1550E- 05
 Total Time=        4.27930     T=       3.18500
 X=  15.68719        XD=       -.11262     XDD=       -1.99968
 Z=  -1.26463        ZD=       -.14521     ZDD=         .27239
 TH=    .03941        THD=       -.00169     THDD=         -.00716
 FBZS= .82969E 06      WRAT=       .011640   VAX=            -.10969

-.1446E  01
-.5187E  01
 .1550E- 05
 Total Time=        4.27430     T=       3.18000
 X=  15.68773        XD=       -.10262     XDD=       -2.00075
 Z=  -1.26390        ZD=       -.14657     ZDD=         .27196
 TH=    .03942        THD=       -.00165     THDD=        -.00717
 FBZS= .82988E 06     WRAT=        .011643    VAX=          -.09975

-.1453E  01
-.4187E  01
 .1431E- 05
 Total Time=        4.26930     T=       3.17500
 X=  15.68822        XD=       -.09261     XDD=       -2.00181
 Z=  -1.26317        ZD=       -.14793     ZDD=         .27152
 TH=    .03943        THD=       -.00162      THDD=        -.00718
 FBZS= .83006E 06       WRAT=      .011645     VAX=           -.08980

-.1432E  01
-.5187E  01
 .1669E- 05
 Total Time=        4.26430     T=       3.17000
 X=  15.68866         XD=      -.08260     XDD=       -2.00287
 Z=  -1.26242         ZD=      -.14928     ZDD=         .27108
 TH=    .03943         THD=     -.00158     THDD=         -.00719
 FBZS= .83024E 06       WRAT=     .011648     VAX=          -.07985

-.1445E  01
-.4187E  01
 .1550E- 05
 Total Time=        4.25930     T=       3.16500
 X=  15.68905         XD=      -.07258     XDD=       -2.00391
 Z=  -1.26167         ZD=      -.15064     ZDD=         .27063
 TH=    .03944         THD=       -.00154    THDD=         -.00720
 FBZS= .83041E 06       WRAT=      .011650    VAX=            -.06990
```

```
-.1447E  01
-.3187E  01
 .1669E- 05
 Total Time=        4.25430     T=        3.16000
 X= 15.68938        XD=        -.06256     XDD=        -2.00494
 Z=  -1.26092       ZD=        -.15199     ZDD=          .27019
 TH=       .03945     THD=       -.00151     THDD=          -.00721
 FBZS=  .83058E  06     WRAT=        .011652    VAX=          -.05994


-.1452E  01
-.4187E  01
 .1431E- 05
 Total Time=        4.24930     T=        3.15500
 X=  15.68967         XD=       -.05253     XDD=        -2.00597
 Z=  -1.26015         ZD=       -.15334     ZDD=          .26974
 TH=      .03946        THD=       -.00147      THDD=          -.00721
 FBZS= .83074E  06       WRAT=      .011655       VAX=          -.04997


-.1452E  01
-.4187E  01
 .1431E- 05
 Total Time=        4.24430     T=        3.15000
 X=  15.68991         XD=       -.04250     XDD=        -2.00698
 Z=  -1.25938         ZD=       -.15469     ZDD=          .26928
 TH=      .03946        THD=      -.00144       THDD=          -.00722
 FBZS= .83090E  06      WRAT=      .011657        VAX=          -.04000


-.1425E  01
-.4187E  01
 .1431E- 05
 Total Time=        4.23930     T=        3.14500
 X=  15.69010       XD=        -.03246     XDD=        -2.00799
 Z=  -1.25861       ZD=        -.15603     ZDD=          .26883
 TH=      .03947      THD=       -.00140      THDD=          -.00723
 FBZS= .83106E  06    WRAT=       .011659       VAX=           -.03003


-.1472E  01
-.2188E  01
 .1311E- 05
 Total Time=        4.23430     T=        3.14000
 X=  15.69023         XD=       -.02242     XDD=        -2.00898
 Z=  -1.25782         ZD=       -.15738     ZDD=          .26837
 TH=      .03948        THD=       -.00136      THDD=          -.00724
FBZS=  .83121E  06      WRAT=       .011661       VAX=          -.02005
```

```
-.1461E  01
-.1187E  01
 .1431E- 05
 Total Time=         4.22930     T=        3.13500
 X=  15.69032        XD=      -.01237     XDD=      -2.00997
 Z=  -1.25703        ZD=      -.15872     ZDD=        .26790
 TH=     .03948       THD=      -.00133    THDD=       -.00725
 FBZS= .83135E  06     WRAT=     .011663    VAX=       -.01006


Test 1 =  -.1437E  01
Test 2 =  -.6187E  01
Test 3 =   .1311E- 05
          State 3 values
          Total Time=         4.22930     T3=        3.13500
          X3=   15.69032       XD3=    -.01237       XDD3=        -2.00997
          Z3=   -1.25703       ZD3=    -.15872       ZDD3=          .26790
          TH3=      .03948       THD3=   -.00133      THDD3=         -.00725
          FBZ3=   .83135E  06    WRAT3=  .011663      VAX3=          -.01006


GAX3=  .1176E  03             (B87)
GAZ3=  .11743E  01            (B86)
  Q1=  .1186E  03             (B192)
  A4= -.2788E- 05
  B4=  .1629E  03             (B194)
  C4= -.1418E  09
          State 4 values
          X4=   15.69032       Z4=        -1.26647     TH4=       .03940   (X3)(B186)(B193)
          Vertical Force at Bow =         .88404E  06                       (B195)
          White Ratio =        .012402                                      (B214)

          Extracting Thrust  =  .31676E  06                                 (B202)
          Ratio of Extracting Thrust to Bollard Thrust IS          6.335
```

thesW5553
Dynamically developed force at the bow o
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